A SHORT ACCOUNT

OF THE

HISTORY OF MATHEMATICS

BY

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PREFACE.

THIS book, most of which is a transcript of some lectures I delivered this year, gives a concise account of the history of mathematics. I have tried to make it as little technical as possible, and I hope that it will be intelligible to any one who is acquainted with the elements of mathematics. Partly to facilitate this, partly to gain space, I have generally made use of modern notation in quoting any results; the reader must therefore recollect that while the matter is the same as that of any writer to whom reference is made his proof is sometimes translated into a more convenient and familiar language.

The history of mathematics begins with that of the Ionian Greeks: the latter were however to some extent indebted to the Egyptians and Phœnicians, and the first chapter is accordingly devoted to a discussion of the mathematical attainments of those races. The subsequent history is divided into that of mathematics under Greek influence, chapters II. to VII.; that of the mathematics of the middle ages and renaissance, chapters VIII. to XIII.; and lastly that of the mathematics of modern times,

chapters XIV. to XIX. In dealing with the subject I have confined myself to giving an account of the lives and discoveries of those mathematicians to whom its development is chiefly due. I should add that I have usually omitted all reference to practical astronomers unless there is some mathematical interest in the theories they proposed.

I hope ultimately to issue a second part which will consist of a biographical index on somewhat the same lines as that published by Poggendorff in 1863. The present volume deals only with the more eminent mathematicians: the index is designed to supplement this by providing as complete a list of mathematicians as is possible. It will give the dates of the birth and death of the writer, a line to say for what he was distinguished, a list of his works, and where possible a reference to some authority where they are treated in detail.

This book is mainly a compilation from existing treatises on the subject; but as these latter are numerous and none of them cover exactly the same ground I dare not suppose it is free from mistakes. I shall thankfully accept notice of any errors or omissions which the reader may detect. A list of the histories dealing with long periods of time is given at the end of this preface, but the only ones of which I have made much use are those there indicated by a star: I generally refer to them by the name of the author only. Monographs on any particular writer or period are usually alluded to in the text or foot-notes. On a subject like this, on large parts of which the authorities are scanty and often conflicting,

there is constantly a difference of opinion among the commentators; but in a popular account I cannot discuss the various views held, and I have therefore in the text only stated that which seems to me on the whole the most reasonable. Any one who will read the works mentioned will be in a position to form an independent judgment.

TRIN. COLL. CAMBRIDGE,
September, 1888.

ERRATUM.

Page 227 line 2, for *infinity* read *continuity*.

ERRATA ET ADDENDA.

Page ix, line 1. Since the publication of this work, the articles here alluded to have been collected into a volume under the title *Greek Geometry from Thales to Euclid*, Dublin, 1889.

Page x, line 8. For 1885—1888 read 1883—1888.

Page x, end of. Add to the list of periodicals *Zeitschrift für Mathematik und Physik edited by Schlömilch, Kahl, and Cantor, and published at Leipzig.*

Page 29, line 5. After $\therefore$ insert *if ρ be the projection of OP on the base of the cylinder, then*

Page 55, lines 27 and 35. For *Proclus* read *Pappus.*

Page 56, line 26. The latest complete edition is that now being issued by J. L. Heiberg. August's edition is confined to the Elements.

Page 81, line 24. Dele $\frac{1}{2}$.

Page 83, line 26. After *by* insert *B. Woodcroft and*

Page 88, line 3. For . *The* read : *the*

Page 110, heading of chapter. For 1543 read 1453.

Page 134, line 29. Dele *vice*.

Page 163, note. Add A life of Roger Bacon by J. S. Brewer is prefixed to the Rolls' series edition of the *Opera Inedita*, London, 1859.

Page 168, line 4 from end. For *Act iv. Sc.* 3 read *Act iv. Sc.* 2.

Page 201, line 10. After *Ludovico* insert *Ferraro, usually known as*

Page 207, line 26. For x^n read x^p.

Page 207, line 27. For *Zeticorum* read *Zeteticorum.*

Page 211, line 15. On more careful consideration I doubt whether Harriot was as much indebted to Vieta as is usually supposed: see the supplement to Rigaud's *Works of Bradley*, Oxford, 1833.

Page 215, line 4. For 2 read 5.

Page 227, line 2. For *infinity* read *continuity.*

Page 230, line 21. After *exclude* insert *all detailed references to*

Page 235, line 8. Before *and* insert *of the theory of equations, the creation of a theory of functions,*

Page 256. Dele the last line.

Page 325, line 2. Add The substance of this and the two following pages is taken from Prof. Williamson's article on the infinitesimal calculus in the ninth edition of the *Encyclopaedia Britannica.* The article also contains a summary of the history of the development of the infinitesimal calculus.

Page 331, second paragraph. I accidentally omitted to allu fact discovered by Gerhardt (*Mathem. Schriften Leibnitze* p. 7) that Leibnitz had at some time copied in his own ha of Newton's tract on analysis by infinite series. A moderi of the Newtonian view of the controversy is given in H. work issued at Leipzig in 1858, of which an English transl published at Cambridge in 1860.

Page 350, line 17. For 8 read 2.

Page 352, line 18. Replace by $g = G\{1 + (\frac{5}{2}m - \epsilon)\sin^2 l\}$

Page 358, line 13. Another volume was published at the sa under the title *Opera Miscellanea*.

Page 358, line 5 from end. For 1760 read 1749.

Page 365. Add at end. The histories of mathematics deal ver with the work produced during the last hundred years; an relied mainly on Marie's *Histoire*, on the references in Pogg *Dictionary*, on the memoirs of the different mathematicia in the ninth edition of the *Encyclopaedia Britannica* (es those on Lagrange, Laplace, Poisson, Hamilton, and Maxw the similar memoirs in the *Penny Cyclopaedia*, and lastly obituary notices in the proceedings of the Royal, the Astron and other Societies: the latter notices contain very full infor about those mathematicians who have died within the la years. Among other works on this period which escaped my tion I ought to mention C. J. Gerhardt's *History of German matics* (concluding chapters); H. Hankel's *Die Entwickelu Mathematik in den letzten Jahrhunderten*, Tübingen, 1885; report by J. L. F. Bertrand on the progress and history of matical analysis.

Page 385, line 5. This requires qualification: see Clerk Maxwell's *tricity*, vol. I. pp. 15 and 28.

Page 399, line 20. I am told that Lavoisier's claims are now adı to be superior to those of Cavendish, though the experime the latter in chemistry are probably the earliest in which i treated as an exact science. Cavendish was educated at Peterl Cambridge.

Page 400, line 2 from end. For *determined the laws of the expansi gases* read *investigated the law of the expansion of a gas changes of temperature*. Dalton's statement of the law was not accurate.

Page 408, line 22. Possibly I ought to have also alluded to Robert W house and William Whewell. I hope shortly to publish a sket the history of mathematics at Cambridge, where I shall deal fully with this question.

Page 414, line 2. For *Berlin* read *Gottingen*.

Page 431, line 21. For *Hellner* read *Hettner*.

WORKS DEALING WITH THE GENERAL HISTORY OF MATHEMATICS, OR LONG PERIODS IN THAT HISTORY.

*ALLMAN, G. J. Articles on *Greek Geometry from Thales to Euclid*, *Hermathena*. Dublin, 1879—1887.

ARAGO, F. J. D. The collected works of Arago, Paris, 1857, contain some seventy *éloges* on different mathematicians of the middle ages and modern times.

ARNETH, A. *Die Geschichte der reinen Mathematik*. Stuttgart, 1852.

*BRETSCHNEIDER, C. A. *Die Geometrie und die Geometer vor Eukleides*. Leipzig, 1870.

BOSSUT, C. *Histoire générale des mathématiques*, 2nd edition, Paris, 1810: 1st ed. translated by J. Bonnycastle, London. 1803.

*CANTOR, M. *Vorlesungen über die Geschichte der Mathematik*†. Vol. I. (to the year 1200 A.D.), Leipzig, 1880.

CHASLES, M. *Aperçu historique sur l'origine et le développement des méthodes en géométrie*. 2nd ed., Paris, 1875.

*DELAMBRE, J. B. J. *Histoire de l'astronomie*. Paris, 1817—27.

DELAMBRE, J. B. J. *Die Arithmetik der Griechen*, ed. by J. J. I. Hoffmann, Maintz, 1817.

DÜHRING, E. *Kritische Geschichte der...Mechanik*. 3rd edition, Leipzig, 1887.

*GOW, J. *A short history of Greek mathematics*. Cambridge, 1884.

GRANT, R. *History of physical astronomy*. 2nd edition, London, 1852.

GÜNTHER, S. *Vermischte Untersuchungen zur Geschichte der mathematischen Wissenschaften*. Leipzig, 1876.

*HANKEL, H. *Zur Geschichte der Mathematik*. Leipzig, 1874.

HEILBRONNER, J. C. *Historia matheseos universae*. Leipzig, 1742.

HELLER, A. *Geschichte der Physik*. Stuttgart, 1882.

*HOEFER, F. *Histoire des mathématiques*. 3rd ed., Paris, 1886.

HUTTON, C. *Dictionary* (2 vols.) and *Tracts* (3 vols.). London, 1812—15.

HUTTON SHAW AND PEARSON. *Phil. Trans. of London, abridged with biographic illustrations*. 18 vols. London, 1809.

ISELY, L. *L'Histoire des mathématiques dans la Suisse française*. Neufchâtel, 1884.

† This valuable work contains a summary of nearly everything that has been written on the subject. There are two other books by the same author: namely, *Mathematische Beiträge zum Kulturleben der Völker*, Halle, 1863; and *Euclid und sein Jahrhundert*, Leipzig, 1867: but most of the matter in these is included in the Lectures.

KAESTNER, A. G. *Geschichte der Mathematik.* 4 vols. Göttingen, 1796.

KLÜGEL, G. S. *Mathematisches Wörterbuch.* 6 vols. Leipzig, 1831

LEWIS, SIR G. C. *The astronomy of the ancients.* London, 1862.

*LIBRI, G. B. I. T. *Histoire des sciences mathématiques en Italie depuis la renaissance.* 4 vols. Paris, 1838—41.

*MARIE, M. *Histoire des sciences mathématiques et physiques.* 12 volumes. Paris, 1885—1888.

MONTUCLA, J. F. *Histoire des mathématiques.* 2nd edition, Paris, 1802.

MURHARD, F. W. A. *Litteratur der mathematischen Wissenschaften.* 4 vols. Leipzig, 1797—1804.

*NESSELMANN, E. H. F. *Die Algebra der Griechen.* Berlin, 1842.

*POGGENDORFF, J. C. *Biographisch-Literarisches Handwörterbuch zur Geschichte der exacten Wissenschaften.* 2 vols. Leipzig, 1863.

QUETELET, L. A. J. *Histoire des sciences mathématiques et physiques chez les Belges.* Bruxelles, 1864.

QUETELET, L. A. J. *Les sciences mathématiques chez les Belges du commencement du* XIX*e siècle.* Bruxelles, 1866.

ROSENTHAL, G. E. *Encyclopaedie der Mathematik.* 4 vols. Gotha, 1794—6.

SUTER, H. *Geschichte der mathematischen Wissenschaften.* 2nd ed., Zürich, 1873.

TANNERY, P. *La géométrie grecque.* Paris, 1887.

TODHUNTER, I. *A history of the calculus of variations during the* XIX*th century.* Cambridge, 1861.

TODHUNTER, I. *A history of the mathematical theory of probability.* Cambridge, 1865.

TODHUNTER, I. *A history of the mathematical theory of attraction.* Cambridge, 1873.

*WOLF, R. *Geschichte der Astronomie.* Munich, 1877.

ZEUTHEN, H. G. *Die Lehre von den Kegelschnitten im Altertum.* Copenhagen, 1886.

There are also three periodicals in which special attention is paid to the history of mathematics. These are the *Bulletino di bibliografia e di storia delle scienze matematische e fisiche* edited by Prince B. Boncompagni, and published at Rome; the *Bulletin des sciences mathématiques et astronomiques* edited by G. Darboux, and published at Paris; and the *Bibliotheca mathematica* edited by G. Eneström, and published at Stockholm.

TABLE OF CONTENTS.

	PAGE
Preface	v
List of authorities	ix
Table of contents	xi

Chapter I. Egyptian and Phoenician Mathematics.

The history of mathematics begins with that of the Ionian Greeks	1
The subsequent history may be divided into three periods	1
The fact that early races were highly civilized is not inconsistent with their being ignorant of mathematics	2
Knowledge of the science of numbers possessed by the Phoenicians	3
Knowledge of the science of numbers possessed by the Egyptians	4
Knowledge of the science of geometry possessed by the Egyptians	7
Probability that the early Greeks learnt from the Egyptians about as much mathematics as is contained in Ahmes' papyrus	10

First Period. Mathematics under Greek influence.

This period begins with the teaching of Thales, circ. 600 B.C., and ends with the capture of Alexandria by the Mohammedans in or about 641 A.D. The characteristic feature of this period is the development of geometry.

Chapter II. The Ionian and Pythagorean Schools.

Section 1. The Ionian School.

THALES, 640—550 B.C.: life of	13
His geometrical discoveries	14
Anaximander, 611—545 B.C.	16

PAGE

Section 2. The Pythagorean School.

Pythagoras, 569—500 B.C.: life of 18
The Pythagorean geometry 23
The Pythagorean theory of numbers 25
Archytas, circ. 400 B.C. 27
His solution of the duplication of a cube 28
Other Greek mathematical schools, circ. 400 B.C. 29
Œnopides. Zeno. Democritus 29

Chapter III. The Schools of Athens and Cyzicus. 420—300 B.C.

Mathematical teachers at Athens prior to 420 B.C. 31
Anaxagoras. Hippias (the quadratrix). Antipho 31
The three problems in which these schools were specially interested . 34
Hippocrates of Chios, circ. 420 B.C. 35
Introduction in geometry of method of reduction . . . 36
The quadrature of certain lunes 37
Plato, 429—348 B.C. 39
Introduction in geometry of rules of analysis 40
Eudoxus, 408—355 B.C. 40
His theorems on the golden section 41
Invention of the method of exhaustions 42
Menæchmus, circ. 340 B.C. 43
Discussion of the conic sections 43
His solution of the duplication of a cube 44
Aristæus. Theætetus 44
Aristotle, 384—322 B.C. 45
Questions on mechanics 45

Chapter IV. The first Alexandrian School. circ. 300—30 B.C.

Foundation of Alexandria 46

Section 1. The third century before Christ.

Euclid, circ. 330—275 B.C. 48
The *Elements* as a text-book of geometry 49
The *Elements* as a text-book of the theory of numbers . . 53
Euclid's other works 55

	PAGE
istarchus, circ. 310—250 B.C.	57
His method of determining the distance of the sun	57
nou. Dositheus. Nicoteles	58
CHIMEDES, 287—212 B.C.: life of	59
His works on plane geometry	61
His works on solid geometry	64
His two papers on arithmetic, and the "cattle problem"	65
His works on the statics of solids and fluids	67
His astronomy	69
The principles of geometry assumed by Archimedes	69
OLLONIUS, circ. 260—200 B.C.	70
His conic sections	71
His other works	73
ntrast between the geometry of Apollonius and that of Archimedes	75
atosthenes, 275—194 B.C.	76

ction 2. The second century before Christ.

ypsicles. Nicomedes (the conchoid). Diocles (the cissoid)	78
rseus. Zenodorus	78
IPPARCHUS, circ. 130 B.C.	79
Foundation of scientific astronomy and of trigonometry	79
ero of Alexandria, circ. 125 B.C.	81
Foundation of scientific engineering and of land-surveying	81
Area of a triangle determined in terms of its sides	82

ction 3. The first century before Christ.

eodosius. Dionysodorus	85

apter V. The second Alexandrian School. 30 B.C.—641 A.D.

ction 1. The first century after Christ.

renus. Menelaus	87
comachus	87
Introduction of the arithmetic current in mediæval Europe	88

ction 2. The second century after Christ.

eon of Smyrna. Thymaridas	88
OLEMY, died in 168	89
His geometry	89
The *Almagest*	90

PAGE

Section 3. The third century after Christ.

Pappus, circ. 280 92
The Συναγωγή, a synopsis of Greek mathematics . . . 92

Section 4. The fourth century after Christ.

Metrodorus. The earliest extant specimens of (rhetorical) algebra . 94
Three stages in the development of algebra 95
DIOPHANTUS, circ. 340 (?) 96
Invention of syncopated algebra in his *Arithmetic* . . 97
The notation, methods, and subject-matter of the work . 97
Subsequent neglect of his discoveries 103
Theon of Alexandria. Hypatia 103

Section 5. The Athenian School (in the fifth century).

Proclus. Damascius. Eutocius 104

Section 6. Roman mathematics (in the sixth century).

Nature of the mathematics read at Rome 105
Boethius, 475—526: life of 106
His works the standard text-books of mediæval Europe . . 107
Cassiodorus. Isidorus 108
The capture of Alexandria, and end of the Alexandrian School. . 109

Chapter VI. The Byzantine School. 641—1543 A.D.

This school served as the channel by which Greek mathematics were introduced into western Europe 110
Hero of Constantinople 110
Psellus. Planudus. Barlaam 111
Argyrus. Nicholas of Smyrna. Pachymores 112
Moschopulus (magic squares) 112

Chapter VII. On systems of numeration.

The methods of counting numbers among primitive races . . 114
Use of the abacus or swan-paa for practical calculation . . . 116
The methods of representing numbers by symbols 119
The Roman and Attic symbols for numbers 119
The Alexandrian or later Greek symbols for numbers . . . 120
Greek arithmetic 120

PAGE

Second Period.

he mathematics of the middle ages and of the renaissance.

his period begins about the sixth century, and may be said to end the invention of analytical geometry and of the infinitesimal calculus. characteristic feature of this period is the creation of modern arith-, algebra, and trigonometry.

pter VIII. The rise of learning in western Europe.

ion 1. *Education in the sixth, seventh, and eighth centuries.*
schools of Charles the Great 123
in, 735—804 124

ion 2. *The Cathedral and Conventual Schools.*
ert (Sylvester II.), died in 1003 126

ion 3. *The rise of the early mediæval universities.*
earliest universities arose during the twelfth century . . 129
three stages through which the mediæval universities passed . 129
-note on the early history of Paris, Oxford, and Cambridge . 130

pter IX. The mathematics of the Arabs.

ion 1. *Extent of mathematics obtained from Greek sources.*
college of scribes 140

ion 2. *Extent of mathematics obtained from the (Aryan) Hindoos.*
A-BHATTA, circ. 530 142
His algebra and trigonometry 142
HMAGUPTA, circ. 640 143
His algebra and geometry 143
SKARA, circ. 1140 145
The *Lilavati* or arithmetic. Origin of Indian symbols . 145
The *Bjita ganita* or algebra 148

PAGE

Section 3. The development of mathematics in Arabia.
Alkarismi or Al-Khwārizmī, circ. 830 150
His *Al-gebr we' l mukabala* 151
His solution of a quadratic equation 152
Tabit ibn Korra, 836—901; solution of a cubic equation . . . 153
Alkhodjandi; statement of a particular case of Fermat's equation . 153
Alhossein. Alkarki. Albategni 154
Albuzjani or Abul-Wafa. Alhazen. Abd-al-gehl 155
Characteristics of the Arabian school 156

Chapter X. The introduction of Arabian mathematical works into Europe.

The eleventh century.
Geber ibn Aphla. Arzachel. 158
The twelfth century.
Adelhard. Ben-Ezra. Gerard. John Hispalensis 158
The thirteenth century.
Leonardo of Pisa, circ. 1175—1230 159
The *Liber abbaci*, 1202 160
The introduction of the arabic numerals into commerce . 160
The introduction of the arabic numerals into science . 161
The mathematical tournament 161
Frederick II., 1194—1250 162
Roger Bacon, 1214—1294 163
Campanus. Jordanus 165
The fourteenth century.
Petrarch 166
The reform of the university curriculum at Vienna 166

Chapter XI. The development of arithmetic to the year 1637.

The Boethian arithmetic 168
Algorism or modern arithmetic 169
Introduced into Italy in the thirteenth century 170

PAGE

Improvements subsequently effected:
(i) The simplification of the fundamental processes, circ. 1450 . 171
(ii) The introduction of signs for addition and subtraction . 174
(iii) The invention of logarithms, 1614 . 174
(iv) The use of decimals, 1619 . 176

Chapter XII. The mathematics of the renaissance. 1450–1637.

Section 1. The development of syncopated algebra and trigonometry.

Regiomontanus, 1436–1476: life of . 179
His *De triangulis* (not printed till 1496) . 181
His *Algorithmus* (not printed till 1534), a syncopated algebra 184
Widman, circ. 1489 . 185
Introduction of symbols + and − into German algebra . 185
Pacioli or Lucas di Burgo, circ. 1500 . 187
His arithmetic and geometry (1494) . 187
Copernicus, 1473–1543 . 190
Record, 1510–1558; introduction of symbol for equality . 190
Stifel, 1486–1567; use of the symbols + and − . 191
Tartaglia, 1500–1559: life of . 193
His solution of a cubic equation (1535) . 194
His arithmetic (1556–60) . 194
Cardan, 1501–1576: life of . 197
His *Ars magna* (1545), the third work printed on algebra . 198
His solution of a cubic equation . 200
Ferrari. Rheticus. Maurolycus . 201
Borrel. Xylander. Commandino. Peletier . 202
Ramus, 1515–1572 . 202
Bombelli, circ. 1570 . 203

Section 2. The development of symbolic algebra.

The two improvements introduced by Vieta . 204
Vieta, 1540–1603: life of . 205
Introduction of symbolic algebra (1591) . 207
His works . 207
Napier, 1550–1617 . 209
Harriot, 1560–1621 . 210
Oughtred, 1574–1660 . 211

Section 3. The origin of the more common symbols in algebra. . 212

PAGE

Chapter XIII. The close of the renaissance. 1586—1637.

Section 1. The development of mechanics and experimental methods.

STEVINUS, 1548—1603: life of . . . . 21[illegible]
Commencement of the modern treatment of statics (1586) . 21[illegible]

GALILEO, 1564—1642: life of . . . . 21[illegible]
Commencement of the science of dynamics . . . 21[illegible]
His astronomy . . . . 22[illegible]

Francis Bacon, 1561—1626 . . . . 22[illegible]

Guldinus, 1577—1643 . . . . 22[illegible]

Section 2. Revival of interest in pure geometry.

KEPLER, 1571—1630: life of . . . . 22[illegible]
His works on geometry (1604 and 1615) introduced the principle of continuity and the method of indivisibles 22[illegible]

Desargues, 1593—1662 . . . . 23[illegible]
Invention of projective geometry (circ. 1639) . . . 23[illegible]

Mathematical knowledge at the close of the renaissance . . . 23[illegible]

PAGE

Chapter XV. History of mathematics from Descartes to Huygens. circ. 1635—1675.

DESCARTES, 1596—1650: life of 23
His views on philosophy 24
His invention of analytical geometry (1637) . . . 24
His algebra, optics, and theory of vortices 24

Cavalieri, 1598—1647 24
The method of indivisibles 24

PASCAL, 1623—1662: life of 24
His geometrical conics 25
The arithmetical triangle 25
Foundation of the theory of probabilities 25
His discussion of the cycloid 25

WALLIS, 1616—1703: life of 2
The *Arithmetica infinitorum* (1656) 2
His analytical conics, algebra, and other works . . 2

FERMAT, 1601—1665: life of 2
His investigations on the theory of numbers . . . 2
His use of infinitesimals 2
Foundation of the theory of probabilities 2

Barrow, 1630—1677: life of 2
The tangent to a curve determined by the angular coefficient 2

HUYGENS, 1629—1695: life of 2
The *Horologium oscillatorium* (1673) 2
The undulatory theory of light 2

Other mathematicians of this time.

De Beaune. Roberval. Dodson. Schooten. Saint-Vincent .
Torricelli. Hudde. Frénicle. Laloubère. Kinckhuysen . .
Courcier. Ricci. Mercator. Brouncker . . . . . .
James Gregory. Wren . . . .
Hooke . . . .
De Sluze. Tschirnhausen. Roemer . . . .

PAGE

Chapter XVI. The life and works of Newton.

The subjects discussed by Newton 281
The life of Newton, and the order of his discoveries . . . 283

Analysis of Newton's works

The *Principia* 303
The *Optics* 321
The essay *On curves of the third degree* 321
The *Quadrature of curves* 323
The theory of fluxions and controversy with Leibnitz . 325
The *Universal Arithmetic* 333
The *Analysis by infinite series* 336
The *Lectiones opticæ* 336
The *Methodus differentialis* 336
The *Analytical geometry* 337

Chapter XVII. Leibnitz and the mathematicians of the first half of the eighteenth century.

Section 1. Leibnitz and the Bernoullis.

LEIBNITZ, 1646—1716; life of 338
His memoirs on the infinitesimal calculus 341
His papers on various mechanical problems . . . 343
Jacob Bernoulli, 1654—1705 345
JOHN BERNOULLI, 1667—1748 346
The younger Bernoullis 347

Section 2. The development of analysis on the continent.

L'Hospital. Parent. Varignon. Saurin 348
Nicole. De Montmort. De Gua. Riccati 349
Fagnano. Viviani 350
Lahire. Rolle 351
CLAIRAUT, 1713—1765 351
D'ALEMBERT, 1717—1783 353
Solution of a partial differential equation of the second order 354

PAGE

Section 3. The English mathematicians of the eighteenth century.

David Gregory 356
TAYLOR, 1685—1731 356
Cotes. Robert Smith. De Moivre 358
MACLAURIN, 1698—1746: life of 359
His geometrical discoveries 360
The *Treatise of fluxions* and propositions on attractions . 361
Stewart. Landen 363

Chapter XVIII. Lagrange, Laplace, and their contemporaries. circ. 1740—1830.

Section 1. The development of analysis and mechanics.

EULER, 1707—1783: life of 366
The *Introductio in analysin infinitorum* 366
The *Institutiones calculi differentialis* 368
The *Institutiones calculi integralis* 368
The *Anleitung zur Algebra* 369
His works on mechanics and astronomy 369
Lambert 371
Bézout. Trembley. Arbogaste. Lhulier 372
Mascheroni 373
Lagrange, 1736—1813: life of 373
His memoirs on various subjects 376
The *Mécanique analytique* 378
The *Théorie des fonctions analytiques* 381
The *Leçons sur le calcul des fonctions* 381
The *Résolution des équations numériques* 382
Characteristics of his work 382
LAPLACE, 1749—1827: life of 383
Invention of the potential and spherical harmonics . . 384
Memoirs on problems in astronomy 385
The *Mécanique céleste* and *Exposition du système du monde* 386
The *Théorie analytique des probabilités* 388
His physical researches 389
Character of Laplace 391
LEGENDRE, 1752—1833 391
His memoirs on attractions 392
The *Théorie des nombres* 393
The *Calcul intégral* and the *Fonctions elliptiques* . . 394

PAGE

Section 6. *Analysis.*

List of recent writers 436
Note on Cauchy 436

Section 7. *Astronomy.*

List of recent writers 438
Note on Leverrier 439

Section 8. *Mathematical physics.*

List of some of the recent writers on the subject 440
Notes on Faraday, Lamé, Foucault, Green, and Maxwell . . 441

Index 447

CHAPTER I.

EGYPTIAN AND PHŒNICIAN MATHEMATICS.

THE history of mathematics cannot with certainty be traced back to any school or period before that of the Ionian Greeks, but the subsequent history may be divided into three periods, the distinctions between which are tolerably well marked. The first period is that of the history of mathematics under Greek influence, this is discussed in chapters II. to VII.: the second is that of the mathematics of the middle ages and the renaissance, this is discussed in chapters VIII. to XIII.: the third is that of modern mathematics, and this is discussed in chapters XIV. to XIX.

Although the history commences with that of the Ionian schools, there is no doubt that those Greeks who first paid attention to it were largely indebted to the previous investigations of the Egyptians and Phœnicians. Our knowledge of the mathematical attainments of those early races, to which this chapter is devoted, is so imperfect that the following brief notes must only be regarded as a summary of the conclusions which seem to me most probable, and the history of mathematics begins with the next chapter.

In the first place we may observe that though all early races which have left records behind them knew something of numeration and mechanics, and though the majority were also acquainted with the elements of land-surveying, yet the rules which they possessed were in general founded only on the results of observation and experiment, and were neither deduced from, nor did they form part of any science. The fact

then that the Egyptians and Phœnicians had reached a high state of civilization does not justify us in assuming that they had studied mathematics.

This remark is illustrated by the history of the Chinese, who, according to some writers*, were familiar with the sciences of arithmetic, geometry, mechanics, optics, navigation, and astronomy nearly three thousand years ago. It is indeed almost certain that the Chinese were then acquainted with several geometrical or rather architectural implements, such as the rule, square, compasses, and level; with a few mechanical machines, such as the wheel and axle; that they knew of the characteristic property of the magnetic needle; and were aware that astronomical events occurred in cycles. But the recent careful investigations of L. A. Sedillot † have shown that the Chinese of that time had made no serious attempt to classify or extend the few rules of arithmetic or geometry which they knew, or to explain the causes of the phenomena with which they were acquainted. The idea that the Chinese had made considerable progress in theoretical mathematics seems to have been due to a misapprehension of the Jesuit missionaries who went to China in the sixteenth century. In the first place they failed to distinguish between the original science of the Chinese and the views which they found prevalent on their arrival; the latter being founded on the work and teaching of Arab missionaries who had come to China in the course of the thirteenth century, and while there introduced a knowledge of spherical trigonometry. In the second place, finding that one of the most important government departments was known as the Board of Mathematics, they supposed that its function was to promote and superintend mathematical studies in the empire. Its duties were really confined to the annual preparation of an almanack, the dates and predictions in which regulated

* See *Études sur l'astronomie indienne et chinoise* by J. B. Biot. Paris, 1862.

† See *Bulletino di Bibliografia e di storia delle scienze matematiche* for May, 1868, p. 161.

many affairs both in public and domestic life. All extant specimens of this almanack are extraordinarily inaccurate and defective. The only geometrical theorem with which as far as I am aware the ancient Chinese were acquainted was that in certain cases (namely when the ratio of the sides was 3 : 4 : 5 or 1 : 1 : $\sqrt{2}$) the area of the square described on the hypotenuse of a right-angled triangle is equal to the sum of the areas of the squares described on the sides. It is barely possible that a few geometrical theorems which can be demonstrated in the quasi-experimental way of superposition were also known to them. In arithmetic their knowledge seems to have been confined to the art of calculation by means of the swan-pan (see p. 116), and the power of expressing the results in writing. Our knowledge of the early attainments of the Chinese, slight though it is, is more complete than in the case of most of their contemporaries. It is thus specially instructive, and serves to illustrate the fact that a nation may possess considerable skill in the applied arts with but little knowledge of the sciences on which those arts are founded.

The only races with whom the Greeks of Asia Minor (amongst whom our history begins) were likely to have come into frequent contact were those inhabiting the eastern littoral of the Mediterranean: and Greek tradition uniformly assigned the special development of geometry to the Egyptians, and that of the science of numbers either to the Egyptians or to the Phœnicians. I will consider their attainments in these subjects separately.

First as to the science of *numbers*. So far as the acquirements of the Phœnicians on this subject are concerned it is impossible to speak with any certainty. The magnitude of the commercial transactions of Tyre and Sidon must however have necessitated a considerable development of arithmetic, to which it is probable the name of science might be properly applied. According to Strabo the Tyrians paid particular attention to the sciences of numbers, navigation, and astronomy; they had we know considerable commerce with their neighbours and

common the Chaldeans, and Strabo says that they regularly supplied the weights and measures used in Babylon. Now the Chaldeans had certainly paid some attention to arithmetic and geometry, as is shown by their astronomical calculations, and whatever was the extent of their attainments in arithmetic it is almost certain that the Phoenicians were equally proficient; while it is probable that the knowledge of the latter, such as it was, was communicated to the Greeks. Personally I am inclined to think that the early Greeks were largely indebted to the Phoenicians for their knowledge of practical arithmetic or the art of calculation. It is perhaps worthy of note that Pythagoras was a Phoenician, as according to Herodotus, but this is more doubtful, Thales was also of that race.

Next as to the mathematics of the Egyptians. Their civilization, and in particular their system of hieroglyphics have been generally recognised as implying that they were fairly proficient in the science of numbers. About twenty years ago a hieratic papyrus, forming part of the Rhind collection in the British Museum, was deciphered, and this has thrown considerable light on their mathematical attainments of the Egyptians. The manuscript was written by a priest named Ahmes somewhere between the years 1700 B.C. and 1100 B.C., and is believed to be itself a copy, with emendations, of an older treatise of about 3400 B.C. The work is called "directions for knowing all dark things," and consists* of a collection of problems in arithmetic and geometry; the answers are given, but in general not the processes by which they are obtained.

The first part deals with the reduction of fractions of the form $2/(2n+1)$ to a sum of fractions whose numerators are each unity: for example, Ahmes states that $\frac{2}{29}$ is the sum of

* See *Ein mathematisches Handbuch der alten Aegypter* [illegible], Leipzig, 1877; and see also [illegible] given in Gow §§ 12–16. [illegible] British Museum another and older papyrus on mathematical subjects which has not yet been deciphered.

Romans on the other hand generally kept the denominator constant and equal to twelve, expressing the fraction (approximately) as so many twelfths. The Babylonians did the same in astronomy, except that they used sixty as the constant denominator; and from them through the Greeks the modern division of a degree into sixty equal parts is derived (see p. 215). Thus in one way or the other the difficulty of having to consider changes in both numerator and denominator was evaded.

Before leaving the question of early arithmetic I should mention that for practical purposes the almost universal use of the abacus or swan-pan rendered it easy to add and subtract, or even to multiply and divide, without any knowledge of theoretical arithmetic. These instruments will be described later in chapter VII.; it will be sufficient here to say that they afford a concrete way of representing a number in the decimal scale, and enable the results of addition and subtraction to be obtained by a merely mechanical process. This, coupled with a means of representing the result in writing, was all that was required in primitive times.

Second as to the science of *geometry*. Geometry is supposed to have had its origin in land-surveying; but while it is difficult to say when the study of numbers and calculation—some knowledge of which was essential in any civilized state—became a science, it is comparatively easy to distinguish between the abstract reasonings of geometry and the practical rules of land-surveying. The principles of land-surveying must have been understood from very early times, but the universal tradition of antiquity asserted that the origin of geometry must be sought in Egypt. That it was not indigenous to Greece and that it arose from the necessity of surveying is rendered the more probable by the derivation of the word from γῆ the earth and μετρέω to measure. Now the Greek geometricians, as far as we can judge by their extant works, always dealt with the science as an abstract one. They sought for theorems which should be absolutely true, and would have argued that to

lines, which is equivalent to determining the trigonometrical ratios of certain angles. The data and the results given agree closely with the measurements of some of the existing pyramids.

Not only are the ideas of trigonometry thus introduced in geometry, but the arithmetical part of the book indicates that Ahmes had some idea of algebraic symbols. The unknown quantity is always represented by the symbol which means a heap; addition is represented by a pair of legs walking forwards, subtraction by a pair of legs walking backwards or by a flight of arrows; and equality by the sign $\angle$. As we shall see in the next chapters the Greeks showed no aptitude for algebra or trigonometry, and it was not until the development of mathematics passed again into the hands of members of a Semitic race that any further progress was made in those subjects. On the other hand all the specimens of Egyptian geometry which we possess deal only with particular numerical problems and not with general theorems; and even if a result was stated as being universally true, it was probably only proved to be so by a wide induction. We shall see later that Greek geometry was from its commencement deductive. There are reasons for thinking that Egyptian geometry made little or no further progress after the date of Ahmes' work: and though for nearly two hundred years after the time of Thales Egypt was recognized by the Greeks as an important school of geometry it would seem that almost from the foundation of the Ionian school the Greeks outstripped their former teachers.

It may be added that Ahmes' book gives us very much that idea of Egyptian mathematics which we should have gathered from statements about it in various Greek and Latin authors, some of whom lived nearly 2000 years later. Previous to its translation several of the more modern commentators were inclined to think that these statements exaggerated the acquirements of the Egyptians, and its discovery must increase the weight to be attached to their testimony.

We know nothing of the applied mathematics (if there were any) of the Egyptians or Phœnicians. The astronomical

FIRST PERIOD.

MATHEMATICS UNDER GREEK INFLUENCE.

This period begins with the teaching of Thales, circ. 600 B.C., *and ends with the capture of Alexandria by the Mohammedans in or about* 641 A.D. *The characteristic feature of this period is the development of geometry.*

It will be remembered that I commenced the last chapter by saying that the history of mathematics might be divided into three periods, namely that of mathematics under Greek influence, that of the mathematics of the middle ages and of the renaissance, and lastly that of modern mathematics. The next four chapters (chapters II.–V.) deal with the history of mathematics under Greek influence: to these it will be convenient to add one (chapter VI.) on the Byzantine school, since through it the results of Greek mathematics were transmitted to Western Europe; and another (chapter VII.) on the systems of numeration which were ultimately displaced by that of the Arabs.

The Ionian School.

The founder of the earliest Greek school of mathematics and philosophy was **Thales**, one of the seven sages of Greece, who was born about 640 B.C.* at Miletus and died in the same town about 550 B.C. The materials for an account of his life are very meagre, and consist of little more than a few anecdotes which have been handed down by tradition. During the early part of his life he was engaged partly in commerce and partly in public affairs; and to judge by two stories that have been preserved, he was then as distinguished for shrewdness in business and readiness in resource as he was subsequently celebrated in science. It is said that once when transporting some salt which was loaded on mules, one of the animals slipping in a stream got its load wet and so caused some of the salt to be dissolved; finding its burden thus lightened it rolled over at the next ford to which it came; to break it of this trick Thales loaded it with rags and sponges, which by absorbing the water made the load heavier and soon effectually cured it of its troublesome habit. At another time, according to Aristotle, when there was a prospect of an unusually abundant crop of olives he got possession of all the olive-presses of the district; and having thus "cornered" them, as I believe the Americans call it, he was able to make his own terms for lending them out and thus realized a large sum. These tales may be apocryphal but it is certain that he must have had some reputation as a man of business and as a good engineer since he was employed to construct an embankment so as to divert the river Halys in such a way as to permit of the construction of a ford.

It was probably as a merchant that Thales first went to Egypt, but during his leisure there he studied astronomy and geometry. He was middle aged when he returned to Miletus;

* Many of the dates in this period are only approximately correct.

these proofs is evidently included in the last, but the early Greek geometers were very timid about generalizing their proofs and were afraid that any additional condition imposed on the triangle might vitiate the general result.

Thales wrote an astronomy, and among his contemporaries was more famous as an astronomer than as a geometrician. It is said that one night when walking out he was looking so intently at the stars that he tumbled into a ditch, on which an old woman exclaimed "how can you tell what is going on in the sky when you can't see what is lying at your own feet?" an anecdote which was often quoted to illustrate the unpractical character of philosophers.

Without going into details it may be mentioned that he taught that a year contained 365 days (and not as was previously reckoned twelve months of thirty days each), he was aware of the sphericity* of the earth, and explained the causes of the eclipses both of the sun and moon. It is well known that he predicted a solar eclipse which took place at or about the time he foretold: the actual date was May 28, 585 B.C. But though this prophecy and its fulfilment gave extraordinary prestige to his teaching, and secured him the name of one of the seven sages of Greece, it is most likely that he only made use of one of the Egyptian or Chaldaean registers which stated that solar eclipses recurred at intervals of 18 years and 11 days.

Anaximander, who was born in 611 B.C. and died in 545 B.C., succeeded Thales as head of the school at Miletus. According to Suidas he wrote a treatise on geometry in which tradition says he paid particular attention to the properties of spheres, and dwelt at length on the philosophical ideas involved in the conception of infinity in space and time. He constructed terrestrial and celestial globes. He is alleged to have introduced the use of the *style* or *gnomon* into Greece. This, in

* According to some recent critics both he and Anaximander believed the earth to be a disc and not a sphere. The statement in the text seems to me to be more probable.

principle, consisted only of a stick stuck upright in a horizontal piece of ground. It was originally used as a sun-dial, in which case it was placed at the centre of three concentric circles such that every two hours the end of its shadow passed from one circle to another. Such sun-dials are found at Pompeii and Tusculum. It is said that he employed it to determine his meridian (presumably by marking the lines of shadow cast by it at sunrise and sunset on the same day, and taking the plane bisecting the angle so formed); and thence, by observing the time of year when the noon-altitude of the sun was greatest and least, he got the solstices; thence, by taking half the sum of the noon-altitudes of the sun at the two solstices, he found the inclination of the equator to the horizon (which determined the latitude of the place), and by taking half their difference, he found the inclination of the ecliptic to the equator. There seems good reason to think that he did actually determine the latitude of Sparta, but it is more doubtful whether he really made the rest of these astronomical deductions.

We need not here concern ourselves further with the successors of Thales. The school he established continued to flourish till about 400 B.C. We know very little of the mathematicians comprised in it, but they would seem to have devoted most of their attention to astronomy. They exercised but slight influence on the further advance of Greek mathematics, which was made almost entirely under the influence of the Pythagoreans. If Thales was the first to direct general attention to geometry, it was Pythagoras, says Proclus, quoting from Eudemus, who "changed the study of geometry into the form of a liberal education, for he examined its principles to the bottom and investigated its theorems in an...intellectual manner"; and it is accordingly to Pythagoras that we must now direct attention.

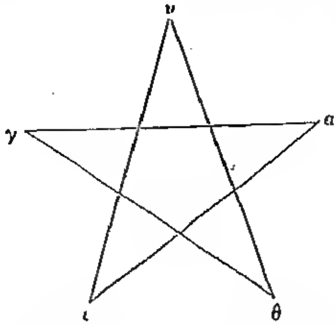
The Pythagorean School.

Pythagoras was born at Samos about 569 B.C. of Tyrian parents, and died in 500 B.C. He was thus a contemporary of Thales. The details of his life are somewhat doubtful, but the following account is I think substantially correct. He studied first under Pherecydes of Syros, and then under Anaximander; by the latter he was recommended to go to Thebes, and there or at Memphis he spent some years. After leaving Egypt he travelled in Asia Minor, and then settled at Samos where he gave lectures but without much success. About 529 B.C. he migrated to Sicily with his mother, and with a single disciple who seems to have been the sole fruit of his labours at Samos. He thence went to Tarentum, but very shortly moved to Croton, a Dorian colony in the south of Italy. Here the schools that he opened were crowded with an enthusiastic audience; citizens of all ranks especially those of the upper classes attended, and even the women broke a law which forbade their going to public meetings and flocked to hear him. Amongst his most attentive auditors was Theano, the young and beautiful daughter of his host Milo, whom, in spite of the disparity of their ages, he married: she wrote a biography of her husband but unfortunately it is lost.

Pythagoras was really a philosopher and moralist, but his philosophy and ethics, as we shall shortly see, rested on a mathematical basis. He divided those who attended his lectures into two classes, the listeners or πυθαγόρειοι and the mathematicians or πυθαγορικοί. In general, a "listener" could, after passing three years as such, be initiated into the second class, to whom alone were confided the chief discoveries of the school. Following the modern usage I shall confine the use of the word Pythagoreans to the latter class.

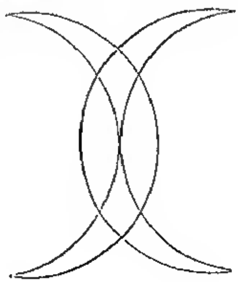
The Pythagoreans formed a brotherhood with all things in common, holding the same philosophical beliefs, engaged in the same pursuits, and bound by oath not to reveal the teaching or secrets of the school. One of the symbols which they used for

purposes of recognition was the pentagram, sometimes also called the triple triangle—τὸ τριπλοῦν τρίγωνον. The penta-



gram is merely a regular re-entrant pentagon; it was considered symbolical of health, and the angles were denoted by the letters of the word ὑγίεια (see p. 36), the diphthong ει being replaced by a θ; it will be noticed that it consists of a single broken line, a feature to which a certain mystical importance* was attached, and on which the theory of the game of "quintan," which is played on a board of that form, depends. Iamblichus, to whom we owe the disclosure of this symbol, tells us how a certain Pythagorean, when travelling, fell ill at a roadside inn where he had put up for the night; he was poor and sick, but the landlord who was a kindhearted fellow nursed him carefully and spared no trouble or expense to relieve his pains.

* Mohammed's signature, said to have been traced in the sand by the point of his scimitar, is another instance of a well-known figure made by a single line. It served as a starting-point for researches by the Arab mathematicians on the geometry of such figures. The subject was subsequently investigated by Euler in his *Solutio problematis ad geometriam situs pertinentis* in the *Mém. de l'Acad. de Berlin* for 1759.



However, in spite of all efforts, the student got worse; feeling that he was dying and unable to make the landlord any pecuniary recompense, he asked for a board on which he inscribed the pentagram-star; this he gave to his host, begging him to hang it up outside so that all the passers-by might see it, and assuring him that he would not then regret his kindness as the symbol on it would ultimately shew. The scholar died and was honourably buried, and the board was duly exposed. After a considerable time had elapsed a traveller one day riding by saw the sacred symbol; dismounting, he entered the inn, and after hearing the story handsomely remunerated the landlord. Such is the anecdote, which if not true, is at least *ben trovato*.

The majority of those who attended the lectures of Pythagoras were only "listeners"; but his philosophy was intended to colour the whole life, political and social, of all his followers. In advocating self-control, government by the best men in the state, strict obedience to legally constituted authorities, and an appeal to eternal principles of right and wrong, he represented a view of society totally opposed to that of the democratic party of that time, and thus naturally most of the brotherhood were aristocrats. It had affiliated members in many of the neighbouring cities, and its method of organization and strict discipline gave it great political power, but like all secret societies it was an object of suspicion to those who did not belong to it. For a short time the Pythagoreans triumphed, but a popular revolt in 501 B.C. overturned the civil government, and in the riots that accompanied the insurrection the mob burnt the house of Milo (where the students lived) and killed many of the most prominent members of the school. Pythagoras himself escaped to Tarentum, and thence fled to Metapontum, where he was murdered in another popular outbreak in 500 B.C.

Though the Pythagoreans as a political society were thus rudely broken up and deprived of their head, they seem to have re-established themselves at once as a philosophical and mathematical society, having Tarentum as their head-quarters. They continued to flourish for a hundred or a hundred and

fifty years after the death of their founder, but they remained to the end a secret society, and we are therefore in ignorance of the details of their history.

Pythagoras himself did not allow the use of text-books, and the assumption of his school was, not only that all their knowledge was held in common and secret from the outside world, but that the glory of any fresh discovery must be referred back to their founder: thus Hippasus (circ. 470 B.C.) is said to have been drowned for violating his oath by publicly boasting that he had added the dodecahedron to the number of regular solids enumerated by Pythagoras. Gradually as the society became more scattered it was found convenient to alter this rule, and treatises containing the substance of their teaching and doctrines were written. The first book of the kind was composed by Philolaus (circ. 410 B.C.), and we are told that Plato contrived to buy a copy of it. We may however say with certainty that during the early part of the fifth cen- tury before Christ they were considerably in advance of th contemporaries, but by the end of that time their more pro nent discoveries and doctrines had become known to outside world, and the centre of intellectual activity was tran. ferred to Athens.

Though it is impossible to separate precisely the discoveries of Pythagoras himself from those of his school of a later date, we know from Proclus that it was Pythagoras who gave geometry that rigorous character of deduction which it still bears, and made it the foundation of a liberal education; and there is good reason to believe that he was the first to arrange the leading propositions of that subject in a logical order. It was also, according to Aristoxenus, the glory of his school that they raised arithmetic above the needs of merchants. It was their boast that they sought knowledge and not wealth, or in the language of one of their maxims, "a figure and a step forwards; not a figure to gain three oboli."

Pythagoras was primarily a moral reformer and practical philosopher, but his system of morality and philosophy was

built up on a mathematical foundation, and it is with his mathematical discoveries alone that I am here concerned. We may perhaps sum up those discoveries by saying that in geometry he himself probably knew and taught the substance of what is contained in the first two books of Euclid, and was acquainted with a few other isolated theorems including some elementary propositions on irrational magnitudes (while his successors added several of the propositions in the sixth and eleventh books of Euclid); but it is thought that many of his proofs were not rigorous, and in particular that the converse of a theorem was frequently assumed without a proof (see Allman's articles). In the theory of numbers he was concerned with four different kinds of problems which dealt respectively with polygonal numbers, ratio and proportion, the factors of numbers, and numbers in series; but many of his arithmetical inquiries, and in particular the questions on polygonal numbers and proportion, were treated by geometrical methods. Knowing that measurement was essential to the accurate definition of form, Pythagoras thought that it was also to some extent the cause of form, and he therefore taught that the foundation of the theory of the universe was to be found in the science of numbers. He was confirmed in this opinion by discovering that the notes sounded by a vibrating string depended (other things being the same) only on the length of the string; and in particular that the lengths which gave a note, its fifth, and its octave were in the ratio $1 : \frac{2}{3} : \frac{1}{2}$. This may have been the reason why music occupied so prominent a position in the exercises of his school. He also believed that the distances of the heavenly bodies from the earth formed a musical progression: hence the phrase 'the harmony of the spheres'. Taking the science of numbers as the foundation of his philosophy he went on to attribute properties to numbers and geometrical figures: for example the cause of colour was the number five; the origin of fire was to be found in the pyramid; a solid body was analogous to the tetrad, which represented matter as composed of the four primary elements, fire, air, earth, and water; and so on. The

A collection of over thirty proofs of Euc. I. 47 was published in *Der Pythagorische Lehrsatz* by Joh. Jos. Ign. Hoffmann, 2nd ed. Mainz, 1821.

(iv) Pythagoras is also credited with the theorems Euc. I. 44, I. 45, and II. 14: on discovering the latter he sacrificed an ox; but as his school had all things in common the liberality was less striking than it seems at first. The Pythagoreans of a later date were aware of the extension given in Euc. VI. 25, and Dr Allman thinks Pythagoras himself was acquainted with it. It will be noticed that Euc. II. 14 is a geometrical solution of the equation $x^2 = ab$.

(v) Pythagoras showed that the plane about a point could be completely filled by equilateral triangles, by squares, or by regular hexagons.

(vi) The Pythagoreans were said to have solved the quadrature of the circle: they stated that the circle was the most beautiful of all plane figures.

(vii) They knew that there were five regular solids inscribable in a sphere which was itself, they said, the most beautiful of all solids.

(viii) From their phraseology in the science of numbers and from other occasional remarks it would seem that they were acquainted with the methods used in the second and fifth books of Euclid, and knew something of irrational magnitudes.

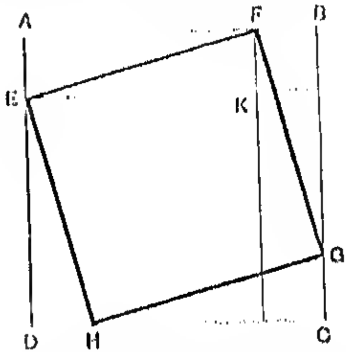
Next as to the theory of numbers*. I have already remarked that in this the Pythagoreans were chiefly concerned with (i) polygonal numbers, (ii) the factors of numbers, (iii) numbers which form a proportion, and (iv) numbers in a series.

Pythagoras commenced his theory of arithmetic by dividing all numbers into even or odd: the odd numbers being termed *gnomons*. An odd number such as $2n+1$ was regarded as the difference of two square numbers $(n+1)^2$ and n^2; and the sum

* See the appendix *Sur l'arithmétique pythagoricienne* to Tannery's *La science hellène*, Paris 1887.

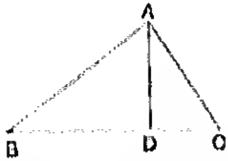
curiosity has been excited to discover what was t
stration of the first of these theorems which was
offered by Pythagoras. It would seem most like
been one of the two following:

(α) Any square $ABCD$ can be split up as in
into two squares BK and DK and two equal rect
and CK: that is, it is equal to the square on FK,
on EK, and four times the triangle AEF.



But if BG, CH, and DE be made equal to AF
easily shown that $EFGH$ is a square, and that th
AEF, BFG, CGH and DHE are equal: thus
$ABCD$ is also equal to the square on EF and
the triangle AEF. Hence the square on EF is e
sum of the squares on FK and EK.

(β) Let ABC be a right-angled triangle, A bein
angle. Draw AD perpe
BC. The triangles ABC
are similar,



$\therefore\ BC : AB :: A$
Similarly $BC : AC :: AC$
Hence
$AB^2 + AC^2 = BC(BD +$

This proof requires a knowledge of Euc. II. 2
VI. 17, with all of which Pythagoras was acquaint

goras was himself acquainted with triangular numbers, but probably not with any other polygonal numbers. A triangular number represents the sum of a number of counters laid in rows on a plane; the bottom row containing n, and each succeeding row one less: it is therefore equal to the sum of the series

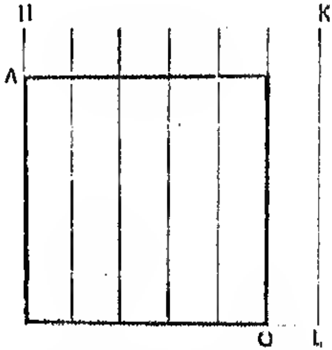
$$n+(n-1)+(n-2)+\ldots+2+1,$$

that is to $\frac{1}{2}n(n+1)$. Thus the triangular number corresponding to 4 is 10. This is the explanation of the language of Pythagoras in the well-known passage in Lucian where the merchant asks Pythagoras what he can teach him. Pythagoras replies "I will teach you how to count." *Merchant*, "I know that already." *Pythagoras*, "How do you count?" *Merchant*, "One, two, three, four—" *Pythagoras*, "Stop! what you take to be four is ten, a perfect triangle, and our symbol." His successors may have treated of polygonal numbers.

As to the work of the Pythagoreans on the factors of numbers we know very little: they classified numbers by comparing them with the sum of their integral factors, terming a number excessive, perfect, or defective according as it was greater than, equal to, or less than the sum of these factors These investigations led to no useful result.

The third class of problems which they considered de with numbers which formed a proportion; these were sumably discussed with the aid of geometry as is done in fifth book of Euclid.

of the gnomons from 1 to $2n+1$ was stated to be a *number*, viz. $(n+1)^2$, its square root was termed a *side*, ducts of two numbers were called *plane*, and if a product exact square root it was termed an *oblong*. A product of numbers was called a *solid number*, and if the three nu were equal a *cube*. All this has obvious reference to geo and the opinion is confirmed by Aristotle's remark that a gnomon is put round a square the figure remains a though it is increased in dimensions. Thus in the figure in which n is taken equal to 5, the gnomon AKC taining 11 small squares) when put round the squa (containing 5^2 small squares) makes a square HL (cont 6^2 small squares). The numbers $(2n^2+2n+1)$, $(2n^2+2$



$(2n+1)$ possessed special importance as representing the tenuse and two sides of a right angled triangle; Prof. thinks that Pythagoras knew this fact before discoveri geometrical proposition Euc. I. 47. A more general e sion for such numbers is (m^2+n^2), (m^2-n^2), and $2mn$; be noticed that Pythagoras assumed $m=n+1$; at a late Archytas and Plato assumed $n=1$; Diophantus kne general rule.

After this preliminary discussion the Pythagoreans ceeded to the four special problems already alluded to.

tion, in polar coordinates (if a be the radius of the cylinder), we have for the equation of the surface described by the semicircle $r=2a\sin\theta$; of the cylinder $r\sin\theta=2a\cos\phi$; and of the cone $\sin\theta\cos\phi=\frac{1}{2}$. These cut in a point such that $\sin^3\theta=\frac{1}{2}$ and $\therefore\ \rho^3=(r\sin\theta)^3=2a^3$. Hence the volume of the cube whose side is ρ is twice that of a cube whose side is a. I mention the problem and give Archytas' construction to illustrate how considerable was the knowledge of the Pythagorean school at that time.

Archytas was one of the most influential citizens of Tarentum, and was made governor of the city no less than seven times. His influence among his contemporaries was very great, and he used it with Dionysius on one occasion to save the life of Plato. He was noted for the attention he paid to the comfort and education of his slaves and of children in the city. Several of the leaders of the Athenian school were among his pupils and friends, and it is believed that much of their work was due to his inspiration. A catalogue of his works is given by Fabricius in *Bib. Graec.* I, p. 833; most of the fragments on philosophy were published by Thomas Gale at Cambridge in 1670.

It would be a great mistake to suppose that Miletus and Tarentum were the only places where Greeks were engaged in laying a scientific foundation for the study of mathematics. These towns represented the centres of chief activity, but there were few cities or colonies of any importance where lectures on philosophy and geometry were not given. Amongst these philosophers who were most influential I may mention Œnopides of Chios (circ. 500 B.C. to 430 B.C.). He devoted himself chiefly to astronomy, but he studied geometry in Egypt, and is credited with the solution of the two problems, (i) to draw a straight line from a given external point perpendicular to a given straight line (Euc. I. 12), and (ii) at a given point to construct an angle equal to a given angle (Euc. I. 23).

Another important centre was at Elea in Italy. The members of the *Eleatic School* were famous for the difficulties they

raised in connection with questions that required the use of infinite series, such for example as the well-known paradox of Achilles and the tortoise, enunciated by Zeno, born in 495 B.C. and died in 435 B.C., one of their most prominent members. Zeno argued that if Achilles ran ten times as fast as a tortoise, yet if the tortoise had (say) 1000 yards start it could never be overtaken: for when Achilles had gone the 1000 yards, the tortoise would still be 100 yards in front of him; by the time he had covered these 100 yards, it would still be 10 yards in front of him; and so on for ever: thus Achilles would get nearer and nearer to the tortoise but never overtake it. The fallacy is obvious to anyone who understands the theory of a geometrical progression. The time required to overtake the tortoise can be divided into an infinite number of parts as stated in the question, but these get smaller and smaller, and the sum of them all is a finite time: after the lapse of that time Achilles would be in front of the tortoise. Zeno himself explained the difficulty by asserting that magnitudes were not infinitely divisible. These paradoxes made the Greeks look with suspicion on the use of infinite series, and ultimately led to the invention of the method of exhaustions (see p. 42). Zeno resided for some years at Athens, circ. 455—450 B.C.

The *Atomistic School* having its head-quarters in Thrace was another important centre. Its most famous member was Democritus, born at Abdera in 460 B.C. and died in 370 B.C., who besides his philosophical works wrote on plane and solid geometry, incommensurable lines, perspective, and numbers. These works are all lost.

But though several distinguished individual philosophers may thus be mentioned who during the fifth century lectured at different cities, they mostly seem to have drawn their inspiration from Miletus or Tarentum, and towards the end of the century to have looked to Athens as the intellectual capital of the Greek world: and it is to the Athenian schools that we owe the next great advance in mathematics.

CHAPTER III.

THE SCHOOLS OF ATHENS AND CYZICUS. CIRC. 420—300 B.C.

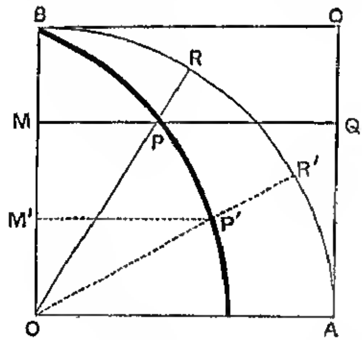
It was towards the close of the fifth century before Christ that Athens first became the chief centre of mathematical studies. Several causes conspired to bring this about. During that century she had become, partly by commerce, partly by appropriating for her own purposes the contributions of her allies, the most wealthy city in Greece; and the genius of her statesmen had made her the centre on which the politics of the peninsula turned. Moreover whatever states disputed her claim to political supremacy her intellectual pre-eminence was admitted by all. There was no school of thought which had not at some time in that century been represented at Athens by one or more of its leading thinkers; and the ideas of the new science, which was being so eagerly studied in Asia Minor and Graecia Magna, had been brought before the Athenians on various occasions.

Amongst the most important of these philosophers who prepared the way for the Athenian school I may mention **Anaxagoras** of Clazomenae (500 to 428 B.C.), who was almost the last philosopher of the Ionian school. He seems to have settled at Athens about 440 B.C., and there taught the results of the Ionian philosophy. Like all members of that school he was much interested in astronomy. He asserted that the sun was larger than the Peloponnesus: this opinion, together with some attempts he had made to explain various physical pheno-

mena which had been previously supposed to be due to the direct action of the gods led to a prosecution for impiety, and he was convicted. While in prison he is said to have written a treatise on the quadrature of the circle. After his release he lived with Pericles, and he died at Athens in 428 B.C.

The sophists can hardly be considered as belonging to the Athenian school any more than Anaxagoras, but like him they immediately preceded and prepared the way for it, so that it is desirable to devote a few words to them. One condition for success in public life at Athens was the power of speaking well; and as the wealth and power of the city increased a considerable number of "sophists" settled there who undertook amongst other things to teach the art of oratory. Many of them also directed the general education of their pupils, of which geometry usually formed a part; and two of them we know made a special study of geometry. These were Hippias of Elis and Antipho.

Hippias of Elis (circ. 420 B.C.) is only known to us by his invention of a curve called the quadratrix* by means of



which an angle could be trisected, or indeed divided in any given ratio. If the radius of a circle rotate uniformly round

* See Part VI. of Dr Allman's papers.

the centre O from the position OA through a right angle to OB, and in the same time a straight line drawn perpendicular to OB move uniformly parallel to itself from the position OA to BC, the locus of their intersection will be the quadratrix.

Let OR and MQ be the positions of these lines at any time; and let them cut in P, a point on the curve. Then $OM : OB =$ arc AR : arc $AB =$ angle AOP : angle AOB. Similarly, if OR' be another position of the radius,

$$OM' : OB = \text{angle } AOP' : \text{angle } AOB.$$

Therefore $\quad OM : OM' = \text{angle } AOP : \text{angle } AOP';$

$$\therefore \quad \text{angle } AOP' : \text{angle } P'OP = OM' : M'M.$$

Hence if the angle AOP be given, and it is required to divide it in any given ratio, it is sufficient to divide OM in that ratio at M', and draw the line $M'P'$; OP' will then divide AOP in the required ratio.

Hippias devised an instrument to construct the curve mechanically; but, as stated before, such devices were objected to by Plato, and rejected by those geometricians who followed him. If OA be taken as the initial line, $OP = r$, the angle $AOP = \theta$, and $OA = a$, we have $\theta : \frac{1}{2}\pi = r \sin\theta : a$, and the equation of the curve is $\pi r = 2a\theta \operatorname{cosec} \theta$.

The other sophist whom I mentioned was **Antipho** (circ. 420 B.C.). He alone amongst the ancients attempted to find the area of a circle by considering it as the limit of an inscribed regular polygon with an infinite number of sides. He began by inscribing an equilateral triangle; on each side in the smaller segment he inscribed an isosceles triangle, and so on *ad infinitum*. I do not know whether he enunciated any general theorems on the subject.

There were probably many other cities in Greece where similar and equally meritorious work was being done, though the record of it has now been lost; and I have only alluded to these two writers in order to give an idea of the kind of investigation which was then going on all over Greece.

The history of the Athenian school begins with the teaching

of Hippocrates about 420 B.C.: the school was established on a permanent basis by the labours of Plato and Eudoxus: and, together with the neighbouring school of Cyzicus, continued to extend on the lines laid down by these three great geometricians until the foundation (about 300 B.C.) of the new university at Alexandria drew most of the talent of Greece thither.

Eudoxus, who was among the most distinguished of the Athenian mathematicians, is also reckoned as the founder of the school at Cyzicus. The connection between this school and that of Athens was very close, and it is now impossible to disentangle their histories. It is said that Hippocrates, Plato, and Theætetus belonged to the Athenian school; while Eudoxus, Menæchmus, and Aristæus belonged to that of Cyzicus. There was always a constant intercourse between the two schools, the earliest members of both had been under the influence either of Archytas or of his pupil Theodorus of Cyrene, and there was no difference in their treatment of the subject, so that they may be conveniently treated together.

Before discussing the work of the geometricians of these schools in detail I may note that they were chiefly concerned with three problems: namely, the duplication of a cube, the trisection of an angle, and the squaring of a circle. Now the first two of these problems (considered analytically) require the solution of a cubic equation: and since a construction by means of circles (whose equations are of the form $x^2+y^2+ax+by+c=0$) and straight lines (whose equations are of the form $\alpha x+\beta y+\gamma=0$) can only be equivalent to the solution of a quadratic or biquadratic equation, the problems are insoluble if we are restricted to lines and circles, i.e. to Euclidean geometry. If the use of the conic sections is permitted, both of these questions can be solved in many ways. The third problem is equivalent to finding a rectangle whose sides are equal respectively to the radius and to the semiperimeter of the circle. These lines have long been known to be incommensurable, but it is only recently that it has been shown

that their ratio cannot be the root of a rational algebraical equation (see Lindemann, *Ueber die zahl* π in *Math. Annalen*, Vol. xx., 1882, p. 213). The Athenians and Cyzicians were thus destined to fail in all three problems, but the attempts to solve them led to the discovery of many new theorems and processes. Besides attacking these problems the later Platonic school collected all the geometrical theorems then known and arranged them systematically. These collections comprised the bulk of the propositions in Euclid, books I.—IX., XI., and XII., together with the elements of conic sections.

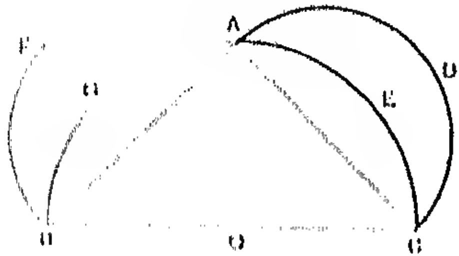
Hippocrates* of Chios (who must be carefully distinguished from his contemporary Hippocrates of Cos, the celebrated physician) was one of the greatest of the Greek geometricians. He was born about 470 B.C. at Chios and began life as a merchant. The accounts differ as to whether he was swindled by the Athenian custom-house officials who were stationed at the Chersonese, or whether one of his vessels was captured by an Athenian pirate near Byzantium: but at any rate somewhere about 430 B.C. he came to Athens to try to recover his property in the law courts. A foreigner was not likely to succeed in such a case, and the Athenians seem only to have laughed at him for his simplicity, first in allowing himself to be cheated, and then in hoping to recover his money. While prosecuting his cause he attended the lectures of various philosophers, and finally (in all probability to earn a livelihood) opened a school of geometry himself. He seems to have been well acquainted with the Pythagorean philosophy, though there is no good authority for the statement that he was ever initiated as a Pythagorean.

He wrote the first elementary text-book of geometry, a text-book on which Euclid's *Elements* was probably founded, and he may therefore be said to have sketched out the lines on which geometry is still taught in English schools. It is supposed that the use of letters in diagrams to describe a figure was made by him or introduced about his time, as he

* See Part III. of Dr Allman's papers.

(α) He commenced by finding the area of a lune contained between a semicircle and a quadrantal arc standing on the same chord. This he did as follows. Let ABC be an isosceles right-angled triangle inscribed in the semicircle $ABOC$ whose centre is O. On AB and AC as diameters describe semicircles as in the figure. Then since $BC^2 = AC^2 + AB^2$ (Euc. I. 47), ∴ by Euc. XII. 2

area ½○ on BC = sum of areas of ½○s on AC and AB.

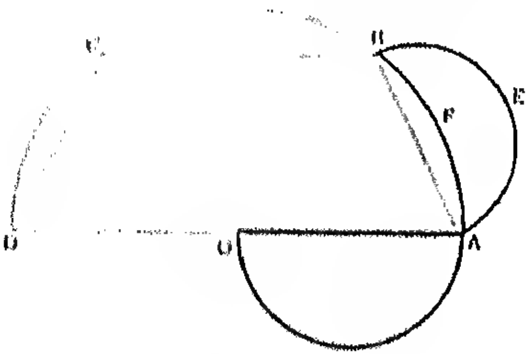


Take away the common parts

∴ area △ABC = sum of areas of lunes $AECD$ and $AFBG$.

Hence the area of the lune $AECD$ is equal to half that of the triangle ABC.

(β) He next inscribed half a regular hexagon $ABCD$ in a



semicircle whose centre was O, and on OA, AB, BC, and CD

as diameters described semicircles of which those on OA and AB are drawn in the figure. Then AD is double any of the lines OA, AB, BC and CD, $\therefore AD^2 = OA^2 + AB^2 + BC^2 + CD^2$,

$\therefore$ area $\frac{1}{2}\odot ABCD$=sum of areas of $\frac{1}{2}\odot$s on OA, AB, BC, and CD.

Take away the common parts

$\therefore$ area trapezium $ABCD = 3$ lune $AEBF + \frac{1}{2}\odot$ on OA.

If therefore the area of this lune is known, so is that of the semicircle described on OA as diameter. According to Simplicius, Hippocrates assumed that the area of this lune was the same as the area of the lune found in proposition (a); if this be so he was of course mistaken, as in this case he is dealing with a lune contained between a semicircle and a sextantal arc standing on the same chord: but it seems probable that Simplicius misunderstood Hippocrates.

Hippocrates also enunciated various other theorems connected with lunes (which are collected in Bretschneider) of which the theorem last given is a typical example. I believe that they are the earliest instances in which areas bounded by curves were determined by geometry.

The other problem to which Hippocrates turned his attention was the duplication of the cube, i.e. to find the side of a cube whose volume should be double that of a given cube. Philoponus says that the Athenians, when suffering from the great plague of eruptive typhoid fever in 430 B.C., consulted the oracle at Delos as to how they could stop it. "Apollo replied that they must double the size of his altar which was in the form of a cube. Nothing seemed more easy, and a new altar was constructed having each of its edges double that of the old one. The god, not unnaturally indignant, made the pestilence worse than before. A fresh deputation was accordingly sent to Delos, whom he informed that it was useless to trifle with him, as he must have his altar exactly doubled. Suspecting a mystery they applied to the geometricians. Plato the most illustrious of them declined the task, but referred them to Euclid who had made a special study of the problem."

The appearance of Euclid's name is accounted for by supposing that some clerk wrote it instead of that of Hippocrates; for the mediæval writers and copyists were not only in the habit of attributing all geometrical theorems to Euclid, but they also constantly confused Euclid the mathematician with the philosopher Euclid of Megara who was a contemporary of Hippocrates. Such is the legend, and at any rate the question was always known as the Delian problem. Eratosthenes gives a similar account of its origin, but with king Minos as the propounder of the problem. Hippocrates reduced the question to that of finding two means x and y between one straight line (a), and another twice as long ($2a$); for if $a : x = x : y = y : 2a$, we have $x^3 = 2a^3$: but he did not succeed in finding these means.

The next great philosopher of the Athenian School was Plato. **Plato** was born at Athens in 429 B.C. and died in 348 B.C.: being wealthy, he was able to devote his time to the pursuits that interested him. He was as is well known a pupil for eight years of Socrates, and much of the teaching of the latter is inferred from Plato's dialogues: after the execution of his master in 399 B.C. Plato left Athens, and travelled for some years. It was during this time that he studied mathematics. He visited Egypt, Megara, Cyrene (where he studied under Theodorus, a distinguished Pythagorean), and Italy. He made a long stay in the latter country, and became very intimate with Archytas the then head of the Pythagorean School, Eurytus of Metapontum, and Timæus of Locri. He returned to Athens about the year 380 B.C., and formed a school of students in a suburban gymnasium called the "Academy." He died in 348 B.C.

Plato, like Pythagoras, was primarily a philosopher, and his philosophy like that of the Pythagoreans was coloured with the idea that the secret of the universe was to be found in number and in form. Hence, as Eudemus says, "he exhibited on every occasion the remarkable connection between mathematics and philosophy." All the authorities agree that,

unlike many later philosophers, he made a study of geometry or some exact science an indispensable preliminary to that of philosophy. The inscription over the entrance to his school ran "Let none ignorant of geometry enter my door," and on one occasion an applicant who knew no geometry is said to have been refused admission as a student.

Plato's position as one of the masters of the Athenian mathematical school rests not so much on his individual discoveries and writings as on the extraordinary influence he exerted on his contemporaries and successors. Thus the objection that he expressed to the use of any instruments other than a ruler and a pair of compasses in the construction of curves was at once accepted as a canon which must be observed in such problems. It is probably due to Plato that subsequent geometricians began the subject with a carefully compiled series of definitions, postulates, and axioms. He also systematized the methods which could be used in attacking mathematical questions, and in particular directed attention to the value of analysis. The analytical method of proof begins by assuming that the theorem or problem is solved, and thence deducing some result: if the result is false the theorem is not true or the problem is incapable of solution: if the result is known to be true, and if the steps are reversible we get (by reversing them) a synthetic proof, but if the steps are not reversible no conclusion can be drawn. Numerous illustrations of the method will be found in any modern text-book on geometry. If the classification of the methods of legitimate induction given by Mill in his work on Logic had been universally accepted, and every new discovery in science had been justified by a reference to the rules there laid down, he would, I imagine, have occupied a position in reference to modern science somewhat analogous to that which Plato occupied in regard to the mathematics of his time.

Of Eudoxus* the third great mathematician of the Athenian

* See Part v. of Dr Allman's papers.

school and the founder of that at Cyzicus, we know very little. He was born in Cnidus in 408 B.C. and died in 355 B.C. Like Plato, he went to Tarentum and studied under Archytas, the then head of the Pythagoreans. Subsequently he went to Egypt where he met Plato, and thence to Cyzicus where he founded the school there. Finally he and his pupils moved to Athens, where he seems to have taken some part in public affairs, and to have practised medicine. The hostility of Plato and his own unpopularity as a foreigner made his position uncomfortable, and he returned to Cyzicus or Cnidus shortly before his death. He died while on a journey to Egypt in 355 B.C.

His mathematical work seems to have been of a high order of excellence. He discovered most of what we now know as the fifth book of Euclid, and proved it in much the same form in which it is there given. He further extended the theorems on what was called "the golden section." The problem to cut a line AB in the golden section, that is, to divide it, say at H, in extreme and mean ratio (i.e. so that $AB : AH = AH : HB$) is solved in Euc. II. 11, and was probably known to the Pythagoreans at an early date.

If we denote AB by l, AH by a, and HB by b, the theorems that Eudoxus proved are equivalent to the following algebraical identities:

(i) $(a+\frac{1}{2}l)^2 = 5(\frac{1}{2}l)^2$. (Euc. XIII. 1.)

(ii) conversely, if (i) be true, and AH be taken equal to a, AB will be divided at H in the golden section. (Euc. XIII. 2.)

(iii) $(b+\frac{1}{2}a)^2 = 5(\frac{1}{2}a)^2$. (Euc. XIII. 3.)

(iv) $l^2+b^2 = 3a^2$. (Euc. XIII. 4.)

(v) $l+a : l = l : a$. This gives another golden section. (Euc. XIII. 5.)

These propositions were subsequently put by Euclid at the commencement of his thirteenth book: but they might have

been equally well placed towards the end of the second book. They are all obvious algebraically, since $l = a + b$ and $a^2 = bl$.

Eudoxus further proved the "method of exhaustions," namely "if from the greater of two unequal magnitudes there be taken more than its half, and from the remainder more than its half, and so on: there will at length remain a magnitude less than the least of the proposed magnitudes." This proposition is given in Euc. x. 1, but in all the modern school editions it is printed at the beginning of the twelfth book. By the aid of this theorem the ancient geometers were able to avoid the use of infinitesimals: the method is rigorous but awkward of application. A good illustration of its use is to be found in the demonstration of Euc. XII. 2, where the proof (as was usual) is completed by a *reductio ad absurdum*, shewing that the square of the radius of one circle is to the square of the radius of another circle as the area of the first circle is to an area which is neither less nor greater than the area of the second circle, and which therefore must be exactly equal to it. Eudoxus applied the principle to shew that the volume of a pyramid (or a cone) is one-third that of the prism (or cylinder) on the same base and of the same altitude (Euc. XII. 7 and 10). Some writers attribute Euc. XII. 2 to him, and not to Hippocrates (see p. 36).

Eudoxus further considered certain curves other than the circle. In particular he discussed some plane sections of the anchor ring, that is, of the solid generated by the revolution of a circle round a straight line lying in its plane. (He assumed that the line did not cut the circle.) A section by a plane through this line consists of two circles: if the plane be moved parallel to itself the sections are lemniscates; when the plane first touches the surface the section is a "figure of eight," generally called Bernoulli's lemniscate, whose equation is $r^2 = a^2 \cos 2\theta$. All this is explained at length in nearly every book on solid geometry. Eudoxus applied the latter curve to explain the apparent progressive and retrograde motions of the planets; but we do not know the method he used. He wrote

a treatise on practical astronomy, in which he adopted a hypothesis previously propounded by Philolaus (409—356 B.C.), and supposed a number of moving spheres to which the sun, moon, and stars were attached, and which by their rotation produced the effects observed. In all he required 27 spheres. As observations became more accurate, subsequent astronomers who accepted his theory had continually to introduce fresh spheres to make the theory agree with the facts. The work of Aratus on astronomy, which was written about 300 B.C. and is still extant, is founded on that of Eudoxus. Eudoxus constructed an orrery.

There seems to be no authority for the statement, which is found in some old books, that Eudoxus studied the properties of the conic sections.

The only other member of these schools requiring special mention is **Menæchmus***, who was a pupil of Plato and of Eudoxus. He was born about 375 B.C. and died about 325 B.C. He acquired great reputation as a teacher of geometry, and was for that reason appointed one of the tutors to Alexander the Great. In answer to Alexander's request to make his proofs shorter, he made the well-known reply, "In the country, sire, there are private and even royal roads, but in geometry there is only one road for all."

Menæchmus was the first to discuss the conic sections which were long called the Menæchmian triads. He divided them into three classes, and investigated their properties, not by taking different plane sections of a fixed cone, but by keeping his plane fixed and cutting it by different cones. He showed that the section of a right cone by a plane perpendicular to a generator is an ellipse, if the cone be acute-angled; a parabola, if it be right-angled; and a hyperbola, if it be obtuse-angled; and he gave a mechanical construction for curves of each class. He also showed how these curves could be used in either of

* See Part VI. of Dr Allman's papers. Dr Allman believes that Menæchmus was one of the successors of Eudoxus as head of the school at Cyzicus.

the two following ways to give a solution of the problem to duplicate a cube. He pointed out that two parabolas having a common vertex, axes at right angles, and such that the latus rectum of the one is double that of the other will intersect in another point whose abscissa (or ordinate) will give a solution: for (using analysis) if the equations of the parabolas are $y^2=2ax$ and $x^2=ay$ they intersect in a point whose abscissa is given by $x^3=2a^3$. He also shewed that the same point could be determined by the intersection of the parabola $y^2=2ax$ and the hyperbola $xy=a^2$. The first of these methods was probably suggested by the form in which Hippocrates had cast the problem, viz. to find x and y so that $a:x=x:y=y:2a$, which gives at once $x^2=ay$ and $y^2=2ax$. In my opinion the solutions show that he was well acquainted with the fundamental properties of these curves, but some writers think that he failed to connect these curves with the sections of the cone which he had discovered.

Of the subsequent members of these schools the only mathematicians of first-rate power were **Aristæus*** and **Theætetus**† whose works are entirely lost. We know however that Aristæus wrote on the five regular solids and on conic sections, and that Theætetus developed the theory of incommensurable magnitudes. The only theorem we can now definitely ascribe to the latter is that given by Euclid in the ninth proposition of the tenth book of the *Elements*: namely, that the squares on two commensurable right lines have one to the other a ratio which a square number has to a square number; and conversely: but the squares on two incommensurable right lines have one to the other a ratio which cannot be expressed as that of a square number to a square number; and conversely. Their successors wrote some fresh text-books on the elements of geometry and the conic sections, introduced problems concerned with finding loci, and efficiently carried out the work commenced by Plato of systematizing the knowledge already acquired.

* See Part VI. of Dr Allman's papers.

† See Part VII. of Dr Allman's papers.

An account of the Athenian school would be incomplete if there were no mention of **Aristotle** who was born at Stagira in Macedonia in 384 B.C. and died at Chalcis in Euboea in 322 B.C. Aristotle however, deeply interested though he was in natural philosophy, was chiefly concerned with mathematics and mathematical physics as illustrations of correct reasoning. A few questions on mechanics which are sometimes attributed to him are of doubtful authority; but though in all probability they are due to another writer, they are interesting partly as shewing that the principles of mechanics were beginning to excite attention, and partly as containing the earliest known employment of letters to indicate magnitudes.

The most instructive parts of the book are the dynamical proof of the parallelogram of forces for the direction of the resultant, and the statement that "if α be a force, β the mass to which it is applied, γ the distance through which it is moved, and δ the time of the motion, then α will move $\frac{1}{2}\beta$ through 2γ in the time δ, or through γ in the time $\frac{1}{2}\delta$"; but the author goes on to say that "it does not follow that $\frac{1}{2}\alpha$ will move β through $\frac{1}{2}\gamma$ in the time δ, because $\frac{1}{2}\alpha$ may not be able to move β at all; for 100 men may drag a ship 100 yards, but it does not follow that one man can drag it one yard." The first part of this statement is correct and is equivalent to the statement that an impulse is proportional to the momentum produced (Newton, Law II.) but the second part is wrong. The author also arrives at the fact that what is gained in power is lost in speed, and therefore that if two weights keep a [weightless] lever in equilibrium they are inversely proportional to the arms of the lever; this, he says, is the explanation why it is easier to extract teeth with a pair of pincers than with the fingers. Other questions asked are, "Why does a projectile ever stop?" and "Why are carriages with large wheels easier to move than those with small?" I ought to add that the book contains some gross blunders, and is not as a whole as able or suggestive as might be inferred from the above extracts.

CHAPTER IV.

THE FIRST ALEXANDRIAN SCHOOL. CIRC. 300—30 B.C.

SECTION 1. *The third century before Christ.* (Euclid. Archimedes.)
SECTION 2. *The second century before Christ.* (Hipparchus. Hero.)
SECTION 3. *The first century before Christ.*

THE earliest attempt to found a university, as we understand the word, was made at Alexandria. Richly endowed, supplied with lecture rooms, libraries, museums, laboratories, gardens, and all the plant and machinery that ingenuity could suggest, it became at once the intellectual metropolis of the Greek race, and remained so for a thousand years. It was particularly fortunate in producing within the first century of its existence three of the greatest mathematicians of antiquity—Euclid, Archimedes, and Apollonius. They laid down the lines on which mathematics were subsequently studied, and, largely owing to their influence, the history of mathematics centres more or less round that of Alexandria until the destruction of the city by the Arabs in 641 A.D.

The city and university of Alexandria were created under the following circumstances. Alexander the Great had ascended the throne of Macedonia in 336 B.C. at the early age of 20, and by 332 B.C. he had conquered or subdued Greece, Asia Minor, and Egypt. Following the plan he adopted whenever a commanding site had been left unoccupied, he founded a new city on the Mediterranean near one mouth of the Nile; and he himself sketched out the ground plan, and arranged for drafts of Greeks, Egyptians, and Jews to be sent to occupy it. The city was intended to be the most magnificent in the world, and the better to secure this, its erection was left in the hands of Dinocrates, the architect of the Temple of Diana at Ephesus.

After Alexander's death in 323 B.C. his empire was divided, and Egypt fell to the lot of Ptolemy, who chose Alexandria as the capital of his kingdom. A short period of confusion followed, but as soon as Ptolemy was settled on the throne, say about 306 B.C., he determined to attract, as far as he was able, learned men of all sorts to his new city; and he at once began the erection of the university buildings on a site of ground immediately adjoining his palace. The university was ready to be opened somewhere about 300 B.C., and Ptolemy who wished to secure for its staff the most eminent philosophers of the time naturally turned to Athens to find them. The great library which was the central feature of the scheme was placed under Demetrius Phalereus, a distinguished Athenian; and so rapidly did it grow that within 40 years it (together with the Egyptian annexe) possessed about 600,000 rolls. The mathematical department was placed under Euclid, who was thus the first, as he was one of the most famous, of the mathematicians of the Alexandrian school.

It happens that contemporaneously with the foundation of this school the information on which our history is based becomes more ample and certain. Many of the works of the Alexandrian mathematicians are still extant: and we have besides an invaluable treatise by Pappus (see p. 92), in which their best known treatises are collated, discussed, and criticized. It curiously turns out that just as we begin to be able to speak with certainty on the subject-matter which was taught, we find our information as to the personality of the teachers becomes uncertain: and we know very little of the lives of the mathematicians mentioned in this and the next chapter, even the dates at which they lived being frequently uncertain.

The third century before Christ.

This century produced the three greatest mathematicians of antiquity, namely Euclid, Archimedes, and Apollonius.

The earliest of these was **Euclid** *. Of his life we know next to nothing. He was of Greek descent, and was born, possibly at Tyre, about 330 B.C.; he died somewhere about 275 B.C. He was well acquainted with the Platonic geometry, but does not seem to have read any of Aristotle's works; and both these facts strengthen the tradition that he was educated at Athens. But whatever may have been his previous training and career he proved to be a most successful teacher when settled at Alexandria. He impressed his own individuality on the teaching of the new university to such an extent, that to his successors and almost to his contemporaries the name Euclid meant (as it does to us) the book or books he wrote, and not the man himself. Some of the mediæval writers went so far as to deny his existence, and with the ingenuity of philologists they explained that the term was only a corruption of ὑκλι a key, and δις geometry. The former word was presumably derived from κλείς. I can only explain the meaning assigned to δις by the conjecture that as the Pythagoreans said that the number two symbolized a line a schoolman may possibly have thought that the representation could be extended to geometry.

From the meagre notices of Euclid which have come down to us we find that the saying that there is no royal road to geometry was attributed to Euclid as well as to Menaechmus; but it is an epigrammatic remark which has had many imitators. "Quel diable pourrait entendre cela?" said a French marquis to Rohault; to which the latter made the apt reply "Ce serait un diable qui aurait de la patience." Euclid is also said to have insisted that knowledge was worth acquiring for its own sake, and Stobaeus (who is a somewhat doubtful authority) tells us that when a lad who had just begun geometry asked "What do I gain by learning all this stuff?" Euclid made

* See the Article *Eucleides* in *Smith's Dictionary of Greek and Roman Biography*, by A. de Morgan, London, 1849; the Article on *Irrational Quantity* in the *Penny Cyclopedia*, by A. de Morgan, London, 1839; and *Litterargeschichtliche Studien über Euklid*, by J. L. Heiberg, Leipzig, 1882.

his slave give the boy some coppers, "Since," said he, "he must make a profit out of what he learns."

According to Pappus Euclid, in making use of the work of his predecessors when writing the *Elements*, dealt most gently with those who had in any way advanced the science: and the Arabian writers, who may perhaps convey to us the traditions of Alexandria, uniformly represent him as a gentle and kindly old man.

Euclid was the author of several works, but his reputation has mainly rested on his *Elements*. This treatise contains a systematic exposition of the leading propositions of elementary geometry (exclusive of conic sections) and of the theory of numbers. It was at once adopted by the Greeks as the standard text-book for the elements of pure mathematics, and on the whole it is probable that it was written for that purpose and not as a philosophical attempt to show that the results of geometry and arithmetic are necessary truths.

The modern text* is founded on an edition prepared by Theon, the father of Hypatia, and is practically a transcript of his lectures at Alexandria (circ. 380 A.D.). There is an older text at the Vatican, and we have besides quotations from the work and references to it by numerous writers of various dates. From these sources we gather that the definitions, axioms, and postulates were rearranged and slightly altered by subsequent editors, but that the propositions themselves are substantially as Euclid wrote them.

As to the matter of the work. The geometrical part is to a large extent a compilation from the works of previous writers. Thus the substance of books I. and II. is probably due to Pythagoras; that of book III. to Hippocrates; that of book V. to Eudoxus; and the bulk of books IV., VI., XI. and XII. to

* Most of the modern text-books are founded on Simson's edition. *Robert Simson*, who was born in 1687 and died in 1768, was professor of mathematics at the university of Glasgow. He wrote several works on ancient geometry: one which is on the Συναγωγή of Pappus is still unpublished.

the later Pythagorean or Athenian schools. But this material was re-arranged, simplified by the omission of obvious deductions (e.g. the proposition that the perpendiculars from the angular points of a triangle on the opposite sides meet in a point was cut out), and in some cases new proofs substituted. The part concerned with the theory of numbers would seem to have been taken from the works of Eudoxus and Pythagoras, except that portion (book x.) which deals with irrational magnitudes. This latter may be founded on the lost book of Theætetus, but much of it is probably original: for Proclus says that while Euclid arranged the propositions of Eudoxus he completed many of those of Theætetus.

The way in which the propositions are proved, consisting of enunciation, statement, construction, proof, and conclusion are due to Euclid: so also is the synthetical character of the work, each proof being written out as a logically correct train of reasoning, but without any clue being given to the method by which it was obtained.

The defects of Euclid as a text-book of geometry have been often stated, and are summed up in de Morgan's article in the *Dictionary of Greek and Roman Biography*. The most prominent are these. (i) The definitions and axioms contain many assumptions which are not obvious, and in particular the so-called axiom about parallel lines is not self-evident*. (ii) No explanation is given as to the reason why the proofs take the form in which they are presented, i.e. the synthetical proof is given but not the analysis by which it was obtained. (iii) There is no attempt made to generalize the results arrived at, e.g. the idea of an angle is never extended so as to cover the case where it is equal to or greater than two right angles†. (iv) The sparing use of superposition as a method of proof,

* It would seem from the recent researches of Grassmann and Riemann that it is incapable of proof.

† The second half of the 33rd proposition in the sixth book, as now printed, appears to be an exception; but it is due to Theon and not to Euclid.

(v) The classification is very imperfect. And (vi) the work is unnecessarily long and verbose.

On the other hand, it still remains in the main a well-arranged chain of geometrical reasoning, proceeding from certain almost obvious assumptions by easy steps to results of considerable complexity. The demonstrations are rigorous, often elegant, and not too difficult for a beginner. Lastly, nearly all the elementary metrical (as opposed to the graphical) properties of space are investigated. The fact that for two thousand years it has been the recognized text-book on the subject raises further a strong presumption that it is not unsuitable for the purpose. During the last few years some determined efforts have been made to displace it in our schools, but the majority of experienced teachers still regard it as the best foundation for geometrical teaching that has yet been published. To this it may be added that some of the greatest mathematicians of modern times, such as Descartes, Pascal, Newton, and Lagrange, advocated its retention as a text-book: and Lagrange said that he who did not study geometry in Euclid would be as one who should learn Latin and Greek from modern works written in those tongues. It must also be remembered that there is an immense advantage in having a single text-book in universal use in a subject like geometry. The unsatisfactory condition of the teaching of geometrical conics in schools is a standard illustration of the evils of using different text-books in such a subject. Some of the objections urged against Euclid do not apply to certain of the recent school editions of his *Elements*. The book has however been generally abandoned on the continent, though apparently with very doubtful advantage to the teaching of geometry.

I do not myself think that all the objections above stated can fairly be urged against Euclid himself. He published two collections of problems generally known as the Δεδομένα or *Data* (containing 95 problems) and the *De divisionibus*. These consist of a graduated series of riders, with hints for their solution; they would afford a sufficient exercise to enable a

student to discover the analysis which led to the proofs given in Euclid, and thus sufficiently answer the second objection. The latter of these two books has only come down to us in a mutilated condition.

I may here add a suggestion thrown out by de Morgan, who is perhaps the most acute of all the modern critics of Euclid. He thinks it likely that the *Elements* were written towards the close of Euclid's life, and their present form represents the first draft of the proposed work, which, with the exception of the tenth book, Euclid did not live to revise. If this opinion be correct, it is probable that Euclid would in his revision have removed the fifth objection.

The geometrical parts of the *Elements* are so well known that I need do no more than allude to them. The first four books and book VI. deal with plane geometry; the theory of proportion (of any magnitudes) is discussed in book V.; and books XI. and XII. treat of solid geometry. Accepting de Morgan's hypothesis that the *Elements* are the first draft of Euclid's proposed work, it is possible that book XIII. is a sort of appendix containing some additional propositions which would ultimately have been put in one or other of the earlier books. Thus as mentioned above (see p. 41) the first five propositions which deal with a line cut in golden section might be added to the second book. The next seven propositions are concerned with the relations between certain incommensurable lines in plane figures (such as the radius of a circle and the sides of an inscribed regular triangle, pentagon, hexagon, and decagon) which are treated by means and as an illustration of the methods of the tenth book. The five regular solids are discussed in the last six propositions. Bretschneider is inclined to think that the thirteenth book is a summary of part of the lost work of Aristaeus: but the illustrations of the methods of the tenth are most probably due to Theaetetus.

Books VII., VIII., IX., and X. of the *Elements* are given up to the theory of numbers. The mere art of calculation or λογιστική was taught to boys when quite young, it was stig-

matized by Plato as childish, and never received much attention from Greek mathematicians; nor was it regarded as forming part of a course of mathematics. We do not know how it was taught, but the abacus certainly played a prominent part in it. The scientific treatment of numbers was called ἀριθμητική, which I have here generally translated as the science of numbers. It had special reference to ratio, proportion, and the theory of numbers. It is with this alone that most of the extant Greek works deal.

In discussing Euclid's arrangement of the subject, we must therefore bear in mind that those who attended his lectures were already familiar with the art of calculation. The system of numeration adopted by the Greeks is described later (see chap. VII.), but it was so clumsy that it rendered the scientific treatment of numbers much more difficult than that of geometry; hence Euclid commenced his mathematical course with plane geometry. At the same time it must be observed that the results of the second book though geometrical in form are all capable of expression in algebraical language, and the fact that numbers could be represented by lines was probably insisted on at an early stage, and illustrated by concrete examples. This method of using lines as symbols for numbers possesses the obvious advantage of giving proofs which are true for all numbers, rational or irrational; it will be noticed that in book II. amongst other things we get a geometrical proof of the distributive and commutative laws, of the rule for multiplication, and finally geometrical solutions of the equations $x^2 + ax - a^2 = 0$ (Euc. II. 11), and $x^2 - ab = 0$ (Euc. II. 14): the solution of the first of these equations is given in the form $\sqrt{a^2 + (\frac{1}{2}a)^2} - \frac{1}{2}a$. The solutions of the equations $x^2 \pm ax - b = 0$ and $x^2 - ax + b = 0$ are given later in Euc. VI. 28 and VI. 29.

The results of the fifth book in which the theory of proportion is considered apply to any magnitudes, and therefore are true of numbers as well as of geometrical magnitudes. In the opinion of de Morgan it is by far the easiest way of treating the theory of proportion on a scientific basis; and it was used

by Euclid as the foundation on which he built the theory of numbers. The theory of proportion given in this book is believed to be due to Eudoxus. The treatment of the same subject in the seventh book is much less elegant, and is supposed to be a reproduction of the Pythagorean teaching. This double discussion of proportion is, as far as it goes, in favour of the conjecture that Euclid did not live to revise the work.

In books VII., VIII., and IX. Euclid discusses the theory of rational numbers. He commences the seventh book with some definitions founded on the Pythagorean notation. In propositions 1 to 3 he shews that if in the usual process for finding the greatest common measure of two numbers the last divisor is unity, the numbers must be prime, and deduces the rule for finding their G.C.M. In propositions 4 to 22 he deals with the theory of fractions, which he founds on the theory of proportion. In propositions 23 to 34 he treats of prime numbers, giving many of the theorems in any modern text-book on algebra (e.g. Todhunter, chap. 52). In propositions 35 to 41 he discusses the least common multiple of numbers, and some miscellaneous problems.

The eighth book is chiefly devoted to numbers in continued proportion, i.e. in a geometrical progression; and the cases where one or more is a product, square, or cube are specially considered.

In the ninth book Euclid continues the discussion of geometrical progressions, and in proposition 35 he enunciates the rule for the summation of a series of n terms, though the proof is only given for the case where n is equal to 4. He also developes the theory of primes, shews that the number of primes is infinite, and discusses the properties of odd and even numbers. He concludes by shewing how to construct a "perfect" number (see p. 27).

In the tenth book Euclid treats of irrational magnitudes; and as the Greeks possessed no symbolism for surds he was forced to adopt a geometrical representation. The first twenty-

one propositions deal generally with incommensurable magnitudes. The rest of the book, namely propositions 22 to 117, is devoted to the discussion of every possible variety of lines which can be represented by $\sqrt{(\sqrt{a} \pm \sqrt{b})}$, where a and b denote commensurable lines. There are twenty-five species of such lines, and that Euclid could detect and classify them all is in the opinion of so competent an authority as Nesselmann the most striking illustration of his genius. It seems almost impossible that this could have been done without the aid of algebra, but the evidence is clear that it was effected by abstract reasoning. In the last proposition (x. 117) the side and diagonal of a square are proved to be incommensurable: this is demonstrated by an *ex absurdo* proof, as it is shown that if they were not so the same number must be both odd and even. Hankel believes that this proof was due to Pythagoras, and was inserted on account of its historical interest. The proposition is also proved, and much more elegantly, in x. 9.

In addition to the *Elements* and the two collections of riders above alluded to (which are extant) Euclid wrote the following books on geometry: (i) an elementary treatise on *conic sections* in four books; (ii) a book on *curved surfaces* (probably chiefly the cone and cylinder); (iii) a collection of *geometrical fallacies*, which were to be used as exercises in the detection of errors; and (iv) a treatise on *porisms* arranged in three books. All of these are lost, but the work on porisms is discussed at such length by Proclus, that some writers have thought it possible to restore it. In particular Chasles in 1860 published what purports to be a reproduction of it, in which will be found the conceptions of cross ratios and projection; in fact those ideas of modern geometry which Chasles and other writers of this century have used so largely. This is brilliant and ingenious, and of course no one can prove that it is not exactly what Euclid wrote, but de Morgan frankly says that he found the statements of Proclus unintelligible, and most of those who read them will, I think, concur in this judgment.

Euclid published two books on optics, namely the *Optics* and the *Catoptrica.* Of these the former is extant, and perhaps the latter too. He commences with the assumption that objects are seen by rays emitted from the eye in straight lines, "for if light proceeded from the object we should not, as we often do, fail to perceive a needle on the floor." The geometry of the books is ingenious. In the *Catoptrica*, if it is the original work written by Euclid, he discusses reflexions in plane, convex, and concave mirrors.

Euclid also wrote the *Phaenomena*, a treatise on geometrical astronomy. It contains references to the work of Autolycus (circ. 330 B.C.), and some book on spherical geometry by an unknown writer. Proclus asserts that Euclid also composed a book on the elements of music: this may refer to the *Sectio canonis* which is by Euclid, and deals with musical intervals.

To these works I may add the following little problem, which is attributed to Euclid. "A mule and a donkey were going to market laden with wheat. The mule said 'if you gave me one measure I should carry twice as much as you, but if I gave you one we should bear equal burdens. Tell me, learned geometrician, what were their burdens.'" It is impossible to say whether the question is genuine, but it would seem to be of about Euclid's time, and is very much the kind of question he would have asked.

Numerous editions of Euclid's works have been issued. The latest complete edition is that by E. F. August printed at Berlin, 1826–9. An accurate English translation of the thirteen books of the *Elements* was published by J. Williamson in 2 vols. Oxford, 1781, and London, 1788, but the notes are unreliable. There is another English translation by Barrow, London and Cambridge, 1660.

It will be noticed that Euclid dealt only with magnitudes, and did not concern himself with their numerical measures, but it would seem from the works of Aristarchus and Archimedes that this was not the case with all the Greek mathematicians of that time. As one of the works of the former

is extant it will serve as another illustration of Greek mathematics of this period.

Aristarchus of Samos, born in 310 B.C. and died in 250 B.C., was an astronomer rather than a mathematician. He asserted, at any rate as a working hypothesis, that the sun was the centre of the universe, and that the earth revolved round the sun. This view, in spite of the simple explanation it afforded of various phenomena, was generally rejected by his contemporaries. But his propositions on the measurement of the sizes and distances of the sun and moon were accurate in principle, and his results were generally accepted as approximately correct (e.g. by Archimedes in his paper on numbers alluded to later, p. 66). A Latin translation of the work containing them was published in London by Wallis in 1691, and another in Paris by d'Urban in 1823. There are 19 theorems, of which I will select the eighth as a typical illustration, because it shows the way in which the Greeks evaded the difficulty of finding the numerical value of surds.

Aristarchus observed the angular distance between the moon when dichotomized and the sun, and found it to be twenty-nine thirtieths of a right angle. It is actually about 89° 21′, but of course his instruments were of the roughest description. He then proceeded to show that the distance of the sun is greater than eighteen and less than twenty times the distance of the moon in the following manner.

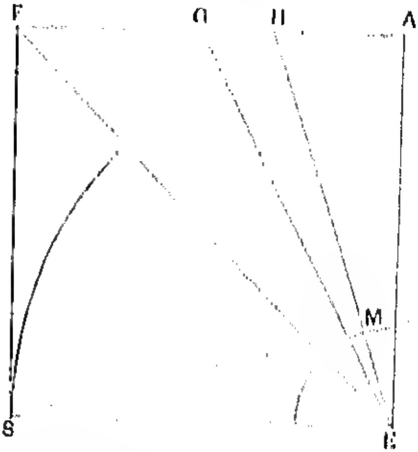
Let S be the sun, E the earth, and M the moon. Then when the moon is dichotomized, that is, when the bright part we can see is exactly a half-circle, the angle between MS and ME is a right angle. With E as centre, and radii ES and EM describe circles, as in the figure on the next page. Draw EA perpendicular to ES. Draw EF bisecting the angle AES, and EG bisecting the angle AEF as in the figure. Let EM (produced) cut AF in H. The angle AEM is by hypothesis $\frac{1}{30}$th of a right angle.

Then angle AEG : angle $AEH = \frac{1}{4}$ rt. $\angle$: $\frac{1}{30}$ rt. $\angle = 15 : 2$,

$$\therefore\ AG : AH\ [= \tan AEG : \tan AEH] > 15 : 2 \ldots\ldots(\alpha).$$

Again $FG^2 : AG^2 = EF^2 : EA^2$ (Euc. VI. 3) $= 2 : 1$ (Euc. I. 47)

$$\therefore\ FG^2 : AG^2 > 49 : 25,$$
$$\therefore\ FG : AG > 7 : 5,$$
$$\therefore\ AF : AG > 12 : 5 \ldots\ldots\ldots\ldots\ldots (\beta$$



Compounding the ratios (α) and (β), $AF : AH > 18 : 1$.
But the triangles EMS and EAH are similar,

$$\therefore\ ES : EM > 18 : 1.$$

I will leave the second half of the proposition to amuse any reader who may care to prove it: the analysis is quite straightforward. In a somewhat similar way Aristarchus found the ratio of the radii of the sun, earth, and moon.

We know very little of Conon, Dositheus, and Nicoteles, the immediate successors of Euclid at Alexandria, except that Archimedes, who was a student at Alexandria, probably shortly after Euclid's death, had a very high opinion of their ability, and corresponded with them. But their reputation has been completely overshadowed by that of Archimedes, whose marvellous mathematical powers have only been surpassed by those of Newton.

Archimedes*, who was probably related to the royal family at Syracuse, was born there in 287 B.C., and died in 212 B.C. He went to the university of Alexandria and attended the lectures of Conon, but as soon as he had finished his studies returned to Sicily, where he passed the remainder of his life. He took no part in public affairs, but his mechanical ingenuity was astonishing, and on any difficulties arising which could be overcome by material means, his advice was generally asked by the government.

Archimedes, like Plato, held that it was undesirable for a philosopher to seek to apply the results of science to any practical use. But whatever might have been his view of what ought to be the case, he did actually introduce a large number of new inventions. The story of the detection of the fraudulent goldsmith, and the use of burning glasses to destroy the ships of the Roman blockading squadron, will recur to most readers. It is not perhaps so generally known that Hiero, who had built so large a ship that he could not launch it off the slips, applied to Archimedes. The difficulty was overcome by means of an apparatus of cogwheels worked by an endless screw, but we are not told exactly how it was used. It is said that Hiero, who was present, exclaimed *Nihil non dicenti Archimedi credam*; to which Archimedes replied, *Da mihi ubi consistam, et terram loco movebo*. Most mathematicians are aware that the Archimedean screw was another of his inventions. It consists of a tube, open at both ends, and bent into the form of a spiral, like a cork-screw. If one end be immersed in water, and the axis of the instrument (i.e. the axis of the cylinder on the surface of which the tube lies) be inclined to the vertical at a sufficiently big angle, and the instrument turned round it, the water will flow along the tube and out at the other end. In order that it may work, the inclination of the axis of the instrument to the vertical must be greater than the pitch of the screw. It was used in Egypt to drain the fields after an inundation of the Nile; and was also

* See *Quaestiones Archimedeae*, by J. L. Heiberg, Hauniae. 1879.

frequently applied to pump water out of the hold of a ship. The story that Archimedes set fire to the Roman ships by means of burning glasses and concave mirrors is not mentioned till some centuries after his death, and is generally rejected: but it is not so incredible as is commonly supposed. The mirror of Archimedes is said to have been made of "a hexagon surrounded by 148 polygons, each of 24 sides"; and Buffon at Paris in 1777 contrived, with the aid of a single composite mirror made on this model, to set fire to wood at a distance of 150 feet, and to melt lead at a distance of 140 feet. This was in April and at a time when the sun was not very bright, so in a Sicilian summer and with several mirrors the deed would be possible, and if the ships were anchored near the town would not be difficult. It is perhaps worth mentioning that a similar device is said to have been used in the defence of Constantinople in 512 A.D., and is alluded to by writers who were either present at the siege or obtained their information from those who were engaged in it. But whatever be the truth as to this story, there is no doubt that Archimedes devised the catapults which kept the Romans, who were then besieging Syracuse, at bay for a considerable time. These were constructed so that the range could be made either short or long at pleasure, and so that they could be discharged through a small loophole without exposing the artillerymen to the fire of the enemy. So effective did they prove that the siege was turned into a blockade, and three years elapsed before the town was taken (212 B.C.).

Archimedes was killed during the sack of the city which followed its capture in spite of the orders, given by the consul Marcellus who was in command of the Romans, that his house and life should be spared. It is said that a soldier entered his study while he was regarding a geometrical diagram drawn in sand on the floor, which was the usual way of drawing figures in classical times. Archimedes told him to get off the diagram, and not spoil it. The soldier insulted at having orders given to him, and ignorant of who the old man was, killed him.

According to another and more probable account, the cupidity of the troops was excited by seeing his instruments constructed of polished brass, which they supposed to be made of gold.

The Romans erected a splendid tomb to Archimedes on which was engraved (in accordance with a wish he had expressed) the figure of a sphere inscribed in a cylinder, in memory of the proof he had given that the volume of a sphere was equal to two-thirds that of the circumscribing right cylinder, and its surface to four times the area of a great circle. Cicero in his *Tusc. Disp.* v. 23 gives a charming account of his efforts (which were successful) to re-discover the tomb in 75 B.C.

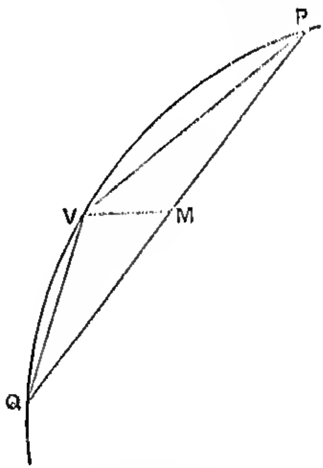
It is difficult to explain in a concise form the works or discoveries of Archimedes, partly because he wrote on nearly all the mathematical subjects then known, and partly because his writings are contained in a series of disconnected monographs. Thus while Euclid aimed at producing systematic treatises which could be understood by all students who had attained a certain level of education, Archimedes wrote a number of brilliant essays addressed chiefly to the most educated mathematicians of the day. The work for which he is perhaps now best known is his mechanics, both of solids and fluids; but he and his contemporaries esteemed his discoveries in geometry of the quadrature of a parabolic area and of a spherical surface, and his rule for finding the volume of a sphere as more remarkable; while at a somewhat later time his numerous mechanical inventions excited most attention.

I. On *plane geometry* the extant works of Archimedes are three in number, namely

(i) *The measure of the circle* in 3 propositions. In the first proposition he proves that the area is the same as that of a right-angled triangle whose sides are equal respectively to the radius a and the circumference of the circle, i.e., the area is equal to $\frac{1}{2}a(2\pi a)$. In the second proposition he shows that $\pi a^2 : (2a)^2 = 11 : 14$ very nearly; and next in the third proposition that π is $> 3\frac{1}{7}$ and $< 3\frac{10}{71}$. These theorems are of course

proved geometrically. To demonstrate the two latter propositions, he inscribes in and circumscribes about a circle regular polygons of 96 sides, calculates their perimeters, and then assumes the circumference of the circle to lie between them. It would seem from the proof that he had some (at present unknown) method of extracting the square roots of numbers approximately.

(ii) *The quadrature of the parabola*, in 24 propositions. He begins this work which was sent to Dositheus by establishing some properties of conics (props. 1—5). He then states correctly the area cut off from a parabola by any chord, and gives a proof which rests on a preliminary mechanical experiment of the ratios of areas which balance when suspended from the arms of a lever (props. 6—17); and lastly he gives a geometrical demonstration of this result (props. 18—24). The latter is of course based on the method of exhaustions, but for brevity I will, in quoting it, use the method of limits.



Let the area of the parabola be bounded by the chord PQ. Draw VM the diameter to the chord PQ, then by a previous

proposition, V is more remote from PQ than any other point in the arc PVQ.

Let the area of the triangle PVQ be denoted by Δ. In the segments bounded by VP and VQ inscribe triangles in the same way as the triangle PVQ was inscribed in the given segment. Each of these triangles is (by a previous proposition of his) equal to $\frac{1}{8}\Delta$, and their sum is therefore $\frac{1}{4}\Delta$. Similarly in the four segments left inscribe triangles; their sum will be $\frac{1}{16}\Delta$. Proceeding in this way the area of the given segment is shown to be equal to the limit of

$$\Delta + \frac{\Delta}{4} + \frac{\Delta}{16} + \dots + \frac{\Delta}{4^n} + \dots,$$

when n is indefinitely large.

The problem is therefore reduced to finding the sum of a geometrical series. This he effects as follows. Let $A, B, C, \dots J, K$, be a series of magnitudes such that each is one fourth of that which precedes it. Take magnitudes $b, c, \dots k$ equal respectively to $\frac{1}{3}B, \frac{1}{3}C, \dots \frac{1}{3}K$. Then

$$B + b = \tfrac{1}{3}A, \quad C + c = \tfrac{1}{3}B, \dots K + k = \tfrac{1}{3}J.$$

Hence $(B + C + \dots + K) + (b + c + \dots + k) = \frac{1}{3}(A + B + \dots + J)$;

but by hypothesis $(b + c + \dots + j + k) = \frac{1}{3}(B + C + \dots + J) + \frac{1}{3}K$;

$$\therefore \quad (B + C + \dots + K) + \tfrac{1}{3}K = \tfrac{1}{3}A;$$

$$\therefore \quad A + B + C + \dots + K = \tfrac{4}{3}A - \tfrac{1}{3}K.$$

That is, the sum of these magnitudes exceeds four times the third of the largest of them by one-third of the smallest of them.

Returning now to the problem of the quadrature of the parabola A stands for Δ, and K is ultimately indefinitely small, therefore the area of the parabolic segment is four-thirds that of the triangle PVQ or two-thirds that of a rectangle whose base is PQ, and altitude the distance of V from PQ.

While discussing the question of quadratures it may be added that in the fifth and sixth propositions of his work on conoids and spheroids he determined the area of an ellipse.

(iii) *On spirals* in 28 propositions. This is on the pro-

perties of the curve now known as the spiral of Archimedes. It was sent to Dositheus at Alexandria accompanied by a letter, from which it seems that Archimedes had previously sent a note of his results to Conon who had died before he had been able to prove them. The spiral is defined by saying that the vectorial angle and radius vector both increase uniformly, hence its equation is $r = c\theta$. Archimedes finds most of its properties, and determines the area inclosed between the curve and two radii vectores. This he does (in effect) by saying, in the language of the infinitesimal calculus, that an element of area is $> \frac{1}{2}r^2 d\theta$ and $< \frac{1}{2}(r+dr)^2 d\theta$: to effect the sum of the elementary areas he gives two lemmas by which he sums (geometrically) the series $a^2 + (2a)^2 + (3a)^2 + \ldots + (na)^2$ (prop. 10), and $a + 2a + 3a + \ldots + na$ (prop. 11).

(iv) In addition to these he wrote a small treatise on *geometrical methods*, and works on *parallel lines*, *triangles*, *the properties of right-angled triangles*, *data*, *the heptagon inscribed in a circle*, and *systems of circles touching one another*; possibly he wrote others too. These are all lost, but it is probable that fragments of four of the propositions in the last mentioned work are preserved in a Latin translation from an Arabic manuscript entitled *The lemmas of Archimedes*.

II. On *geometry of three dimensions* the extant works of Archimedes are as follows.

(i) *The sphere and cylinder* in two books of 60 propositions. Archimedes sent this like so many of his works to Dositheus at Alexandria, but he seems to have played a practical joke on his friends there; for he purposely misstated some of his results "to deceive those vain geometricians who say they have found everything, but never give their proof, and sometimes claim that they have discovered what is impossible." He regarded this work as his masterpiece. It is too long for me to give an analysis of its contents, but I remark in passing that in it he found expressions for the surface and volume of a pyramid, of a cone, and of a sphere, as well as of the figures produced by the revolution of polygons

inscribed in a circle about a diameter of the circle. There are several other propositions on areas and volumes of which perhaps the most striking is the tenth proposition of the second book, namely that "of all spherical segments whose surfaces are equal, the hemisphere has the greatest volume." In the second proposition of the second book he enunciated the remarkable theorem that a line of length a can only be divided so that $a-x : b = c^2 : x^2$ (where b is given, and $c = \frac{2}{3}a$) if $a-c$ be greater than b. That is to say the cubic equation $x^3 - ax^2 + \frac{4}{9}a^2b = 0$ can only have a real and positive root if a be greater than $3b$. This proposition was required to complete his solution of the problem to divide a given sphere by a plane so that the volumes of the segments should be in a given ratio. (See Cantor, pp. 265—271.) One very simple cubic equation occurs in the *Arithmetic* of Diophantus, but with that exception no such equation appears again in the history of mathematics for more than a thousand years.

(ii) A work on quadrics of revolution called *Conoids and spheroids* in 40 propositions (sent to Dositheus in Alexandria) most of which is devoted to an investigation of their volumes.

And (iii), he also wrote a treatise on the thirteen *semi-regular polyhedrons*, that is, solids contained by regular but dissimilar polygons. This is lost.

III. He wrote two papers on *arithmetic*: one on the principles of numeration, addressed to Zeuxippus, which is now lost; and another addressed to Gelon called Ψαμμίτης (*the sand-reckoner*) in which he meets an objection which had been urged against his first paper. The object of the first paper had been to suggest a convenient system by which numbers of any magnitude could be represented; and it would seem that some philosophers at Syracuse had doubted whether it could be used. He says people talk of the sand on the Sicilian shore as something beyond the power of calculation, but he can estimate it, and further he will illustrate the power of his method by finding a superior limit to the number of grains of sand which would fill the whole universe, i.e. a

sphere whose centre is the earth, and radius the distance of the sun. He begins by saying that in ordinary Greek nomenclature it was only possible to express numbers from 1 up to 10^8: these are expressed in what he says he may call units of the first order. If 10^8 be termed a unit of the second order, any number from 10^8 to 10^{16} can be expressed as so many units of the second order plus so many units of the first order. If 10^{16} be a unit of the third order any number up to 10^{24} can then be expressed; and so on. Assuming that 10,000 grains of sand occupy a sphere whose radius is not less than $\frac{1}{80}$th of a finger breadth, and that the diameter of the universe is not greater than 10^{10} stadia, he finds that the number of grains of sand required to fill the universe is less than 10^{63}. The essay was probably merely a scientific curiosity. The Greek system of numeration with which we are acquainted had only recently been introduced, probably at Alexandria, and was sufficient for all the purposes for which the Greeks then required numbers: and Archimedes used that system in all his papers. On the other hand it is most likely that Archimedes and Apollonius had some symbolism based on the decimal system for their own investigations; and it is very possible that it was the one here sketched out.

To these two arithmetical papers, I may add the following celebrated problem which he sent to the Alexandrian mathematicians. The sun had a herd of bulls and cows, all of which were either white, grey, dun, or piebald: the number of piebald bulls was less than the number of white bulls by 5/6ths of the number of grey bulls, it was less than the number of grey bulls by 9/20ths of the number of dun bulls, and it was less than the number of dun bulls by 13/42ths of the number of white bulls: the number of white cows was 7/12ths of the number of grey cattle (bulls and cows), the number of grey cows was 9/20ths of the number of dun cattle, the number of dun cows was 11/30ths of the number of piebald cattle, and the number of piebald cows was 13/42ths of the number of white cattle. The problem was to find the

composition of the herd; it is indeterminate, but the solution in lowest integers is

white bulls,	10,366,482.	white cows,	7,206,360.
grey bulls,	7,460,514.	grey cows,	4,893,246.
dun bulls,	7,358,060.	dun cows,	3,515,820.
piebald bulls,	4,149,387.	piebald cows,	5,439,213.

In the classical solution, attributed to Archimedes, these numbers are multiplied by 80.

Nesselmann believes, from internal evidence, that the problem has been falsely attributed to Archimedes. It certainly is very unlike his extant work, but it was universally attributed to him among the ancients, and is generally thought to be genuine, though possibly it has come down to us in a modified form. It is in verse, and a later copyist has added the additional conditions that the sum of the white and grey bulls shall be a square number, and the sum of the piebald and dun bulls a triangular number.

It is perhaps worthy of note that in the enunciation the fractions are still represented as a sum of fractions whose numerators are unity: thus Archimedes wrote $1/7 + 1/6$ instead of $13/42$ (see p. 5).

IV. His works on *mechanics* comprise

(i) His *Mechanics*. This is a work on statics with special reference to the equilibrium of plane laminæ and to properties of their centres of gravity; it consists of 25 propositions in two books. In the first part of book I. most of the elementary properties of the centre of gravity are proved (props. 1—8); and in the remainder of book I. (props. 9—15) and in book II. the centres of gravity of a variety of plane areas, such as parallelograms, triangles, trapeziums, and parabolic areas, are determined.

(ii) A treatise on *levers* and perhaps on all the mechanical machines. The book is lost, but we know from Pappus that it contained a discussion of how a given weight could be moved with a given power. It was in this work probably that Archimedes discussed the theory of a certain compound pulley

consisting of three or more simple pulleys which he had invented, and which was used in some public works in Syracuse. It is well known that he boasted that if he had but a fixed fulcrum to act on he would move the whole earth (see p. 59); and a late writer says that he added he would do it by using a compound pulley.

(iii) A work on *floating bodies*, containing 19 propositions in two books. This was the first attempt to apply mathematical reasoning to hydrostatics. The story of the manner in which his attention was directed to the subject is told by Vitruvius. Hiero, the king of Syracuse, had given some gold to a goldsmith to make into a crown. The crown was delivered, made up, and of the proper weight, but it was suspected that the workman had appropriated a considerable proportion of the gold, replacing it by an equal weight of silver. Archimedes was thereupon consulted. Shortly afterwards, when in the public baths, he noticed that his body was pressed upwards by a force which increased the more completely he was immersed in the water. Recognizing the value of the observation, he rushed out just as he was, and ran home through the streets, shouting εὕρηκα, εὕρηκα, "I have found it," "I have found it." There (to follow a later account) on making accurate experiments he found that when equal weights of gold and silver were weighed in water they no longer appeared equal: each seemed lighter than before by the weight of the water it displaced, and as the silver was more bulky than the gold its weight was most diminished. Hence if on a balance he weighed the crown against an equal weight of gold, and then immersed the whole in water, the gold would outweigh the crown if any silver had been used in its construction. Tradition says that the goldsmith was found to be fraudulent.

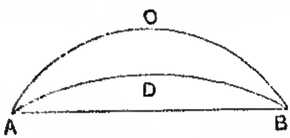
Archimedes began the work by proving that the surface of a fluid at rest is spherical, the centre of the sphere being at the centre of the earth. He then proved that the pressure of the fluid on a body, wholly or partially immersed, is equal to the weight of the fluid displaced; and thence found the position

of equilibrium of a floating body, which he illustrated by spherical segments and paraboloids of revolution floating on a fluid. Some of the latter problems involve geometrical reasoning of great complexity.

The following is a fair specimen of the questions considered. "A solid in the shape of a paraboloid of revolution of height h and latus rectum $4a$ floats in water, with its vertex immersed and its base wholly above the surface. If equilibrium be possible when the axis is not vertical, then the density of the body must be less than $(h-3a)^2/h^2$" (book II. prop. 4). If it is recollected that Archimedes had not the use either of trigonometry or of analytical geometry, this and similar propositions will serve as an illustration of his powers of analysis.

V. We know both from occasional references in his works and from remarks by other writers that he was largely occupied in *astronomical observations*. He wrote a book, Περὶ σφειροποιΐας, on the construction of a celestial sphere, which is lost; and he constructed a sphere of the stars, and an orrery. These after the capture of Syracuse were taken by Marcellus to Rome, and were preserved as curiosities for at least two or three hundred years.

This mere catalogue of his works will show how wonderful were his achievements; but no one who has not actually read some of his writings can form a just appreciation of his extraordinary ability. This will be still further increased if we recollect that the only principles used by Archimedes, in addition to those contained in Euclid's *Elements* and *Conic sections*, are that of all lines like ACB, ADB, ... connecting two points A and B, the straight line is the shortest, and of the curved lines, the inner one ADB is shorter than the outer one ACB; together with two similar statements for space of three dimensions.



The value of his work may also be surmised from the fact that all text-books on statics rested on his theory of the lever

until the publication of Stevinus' work in 1586: and no distinct advance was made in the theory of hydrostatics until Stevinus in the same work investigated the laws which regulate the pressure of fluids (see p. 217).

In the old and mediæval world Archimedes was unanimously put as the first of mathematicians: and in the modern world there is no one but Newton who can be compared with him. Perhaps the best tribute to his fame is the fact that those writers who have spoken most highly of his work and ability are those who have been themselves the most distinguished men of their own generations.

The latest and best edition of the extant works of Archimedes is that by J. L. Heiberg, in 3 vols., Leipzig, 1881. There is a very good German translation of them by Gutenäcker, Würzburg, 1828.

The third great mathematician of this century was **Apollonius*** of Perga, who is chiefly celebrated for having produced a systematic treatise on the conic sections which not only included all that was previously known about them but immensely extended the knowledge of these curves. This work was at once accepted as the standard text-book on the subject, and completely superseded the previous treatises of Menaechmus, Aristaeus, and Euclid which up to that time had been in general use.

We know very little of Apollonius himself. He was born about 260 B.C. and died about 200 B.C. He studied in Alexandria for many years and probably lectured there; he is represented by Pappus as "vain, jealous of the reputation of others, and ready to seize every opportunity to depreciate them." It is curious that while we know next to nothing of his life, or of that of his contemporary Eratosthenes, yet their nicknames, which were respectively *epsilon* and *beta*, have come down to us. Dr Gow has ingeniously suggested that the lecture rooms at Alexandria were numbered, and

* See *Litterargeschichtliche Studien über Euklid*, by J. L. Heiberg, Leipzig, 1882.

that they always used the rooms numbered 5 and 2 respectively.

Apollonius spent some years at Pergamum in Pamphylia, where a university had recently been established and endowed in imitation of that at Alexandria. There he met Eudemus and Attalus to whom he subsequently sent each book of his conics as it came out with an explanatory note. He returned to Alexandria, and lived there till his death, which was nearly contemporaneous with that of Archimedes.

In his great work on *Conic sections* he so thoroughly investigated the properties of these curves that he left but little for his successors to add. But his proofs are long and involved, and I think most readers will be content to accept a short analysis of his work, and the assurance that his demonstrations are valid.

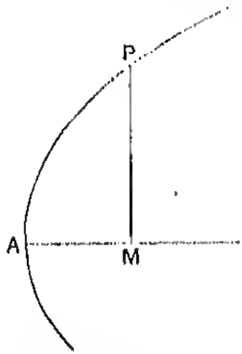
The treatise contained about 400 propositions and was divided into eight books; we have the Greek text of the first four of these, and we also possess copies of the commentaries by Pappus and Eutocius on the whole work. An Arabic translation was made in the ninth century of the first seven books, which were the only ones then extant; we have two manuscripts of this version. The eighth book is lost.

In the letter to Eudemus which accompanied the first book Apollonius says that he undertook the work at the request of Naucrates, a geometrician who had been staying with him at Alexandria, and though he had given some of his friends a rough draft of it, he had preferred to revise it carefully before sending it to Pergamum. In the note which accompanied the next book, he asks Eudemus to read it and communicate it to others who can understand it, and in particular to Philonides a certain geometrician whom the author had met at Ephesus.

The first four books deal with the elements of the subject, and of these the first three are founded on Euclid's previous work (which was itself based on the earlier treatises by Menaechmus and Aristaeus). Heracleides asserts that much of the matter in these books was stolen from an unpublished

work of Archimedes, but a critical examination by Heiberg has shown that this is not the case.

Apollonius begins by defining a cone on a circular base. He then investigates the different plane sections of it, and shows that they are divisible into three kinds of curves which he calls ellipses, parabolas, and hyperbolas. He then proves the proposition that the ratio (in the usual notation) $PM^2 : AM \, . \, MA'$ is constant in an ellipse or hyperbola, and the ratio $PM^2 : AM$ is constant in a parabola. These are the characteristic properties on which almost all the rest of the work is based.



He next shows that if A be the vertex, l the latus rectum, and if AM and MP be the abscissa and ordinate of any point on a conic, then MP^2 is less than, equal to, or greater than $l \, . \, AM$ according as the conic is an ellipse, parabola, or hyperbola; hence the names which he gave to the curves and by which they are still known.

He had no idea of the directrix and was not aware that the parabola had a focus, but with the exception of the propositions which involve these his first three books contain most of the propositions which are in any modern text-book.

In the fourth book he develops the theory of lines cut harmonically, and treats of the points of intersection of systems of conics. In the fifth book he commences with the theory of maxima and minima, and applies it to find the centre of curvature at any point of a conic and the evolute of the curve, and discusses the number of normals which can be drawn from a point to a conic. In the sixth book he treats of similar conics. The seventh and eighth books were given up to a discussion of conjugate diameters, the latter of these was conjecturally restored by Halley in 1710 A.D. when Savilian professor at Oxford.

The verbose and tedious explanations make the book repulsive to most modern readers; but the logical arrangement and reasoning are unexceptional, and it has been not unfitly described as the crown of Greek geometry. It is the work on which the reputation of Apollonius rests, and it earned for him the name of "the great geometrician."

Besides this immense treatise he wrote numerous shorter works, of which the following list contains all about which we now know anything. He was the author of a work on the following problem. Given two co-planar straight lines Aa and Bb, drawn through fixed points A and B; to draw a line Oab from a given point O outside them cutting them in a and b, so that Aa shall be to Bb in a given ratio. He reduced the problem to a large number of separate cases and gave an appropriate solution, with the aid of conics, for each case. He also wrote a treatise *De sectione spatii* (restored by E. Halley in 1706) on the same problem under the condition that the rectangle $Aa \, . \, Bb$ was given. He wrote another entitled *De sectione determinata* (restored by R. Simson, Glasgow, 1749), dealing with problems such as to find a point P in a given straight line AB such that $PA^2 : PB$ shall be in a given ratio. He wrote another *De tactionibus* on the construction of a circle which shall touch three given circles (restored by Vieta, see p. 206). Another work was his *De inclinationibus* (restored by M. Ghetaldi, Venice, 1607) on the problem to draw a line so that the intercept between two given lines, or the circumferences of two given circles, shall be of a given length. He was also the author of a treatise in three books on plane loci *De locis planis* (restored by R. Simson in 1746), and of another on the *regular solids.* And lastly he wrote a treatise on *unclassed incommensurables*, being a commentary on the tenth book of Euclid. It is believed that in one or more of the lost books he used the method of conical projections.

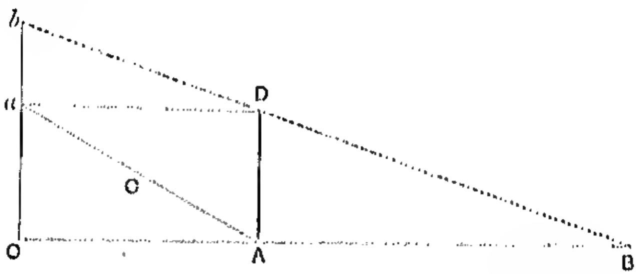
Besides these geometrical works he wrote on the *methods of arithmetical calculation.* This would be of great value if we could obtain a copy, but we know nothing of its contents be-

yond the fact that he pointed out that a decimal system of notation, involving only nine* symbols, would vastly facilitate numerical multiplications. He suggested a system of numeration similar to that proposed by Archimedes (see p. 65), but proceeding by tetrads instead of octads, and described a notation for it. It will be noticed that our modern notation goes by hexads, a million $= 10^6$, a billion $= 10^{12}$, a trillion $= 10^{18}$, &c.

He was interested in astronomy, and wrote a book on the *stations and regressions of the planets*, of which Ptolemy made some use in writing the *Almagest*. He also wrote a treatise on the use and theory of the screw in statics.

This is a long list, but I should suppose that most of these works were short tracts on special points.

Like so many of his predecessors, he too gave a construction for finding two mean proportionals between two given lines, and thereby duplicating the cube. It was as follows. Let OA, Oa, be the given lines. Construct a rectangle $OADa$, of which they are adjacent sides. Bisect Aa in C. Then if with C as centre we can describe a circle cutting OA in B and cutting



Oa in b, so that BDb shall be a straight line, the problem is effected. For it is easily shown that

$$OB \,.\, AB + OA^2 = CB^2.$$

* It is barely possible that he used a dot . to indicate the absence of a symbol in the same way as we use the symbol 0.

Similarly $$Ob \,.\, ab + Ca^2 = Cb^2.$$
Hence $$OB \,.\, AB = Ob \,.\, ab.$$
That is $$OB : Ob = ab : AB.$$
But, by similar triangles,
$$aD : ab = OB : Ob = AB : AD.$$
Therefore $$OA : ab = ab : AB = AB : Oa.$$
It is impossible to construct the circle by Euclidean geometry, but Apollonius gave a mechanical way of describing it.

The solution of Apollonius is that given by most Arabic writers. In the *Tarikhu-l-hukama* the solution is prefaced by the following account of the origin of the problem which is a curious corruption of that given above on p. 38. "Now in the days of Plato a plague broke out among the children of Israel. Then came a voice from heaven to one of their prophets, saying, 'let the size of the cubic altar be doubled, and the plague will cease'; so the people made another altar like unto the former, and laid the same by its side. Nevertheless the pestilence continued to increase. And again the voice spake unto the prophet, saying, 'they have made a second altar like unto the former, and laid it by its side, but that produces not the duplication of the cube.' Then applied the Israelites to Plato, the Grecian sage, who spake to them, saying, 'ye have been neglectful of the science of geometry, and therefore hath God chastised you, since geometry is the most sublime of all the sciences.' Now, the duplication of a cube depends on a rare problem in geometry, namely..." And then follows the solution of Apollonius.

The best collection of the extant works of Apollonius are the editions by E. Halley, Oxford, 1706 and 1710.

In one of the most brilliant passages of his *Aperçu historique* Chasles remarks that while Archimedes and Apollonius are the most able geometricians of the old world, their works are distinguished by a contrast which runs through the whole subsequent history of geometry. Archimedes in attacking the problem of the quadrature of curvilinear areas

laid the foundation of the geometry which rests on measurements; this naturally gave rise to the infinitesimal calculus, and in fact the method of exhaustions as used by Archimedes does not differ in principle from the method of limits as used by Newton. Apollonius, on the other hand, in investigating the properties of conic sections by means of transversals involving the ratio of rectilineal distances and of perspective, laid the foundations of the geometry of form and position.

Among the contemporaries of Archimedes and Apollonius I may mention **Eratosthenes**. Born at Cyrene in 275 B.C. he was educated at Alexandria and Athens, and was at an early age entrusted with the care of the university library at Alexandria, a post which he occupied till his death in 194 B.C. He was the Admirable Crichton of his age. His pre-eminence in the five events to which amateur athletics were then confined, was so marked that he was popularly named *pentathlus*. He was something of a poet and wrote on various literary subjects. In science he was chiefly interested in astronomy and geodesy, and he constructed the astronomical instruments which were used for some centuries at the university. He introduced the Julian calendar, in which every fourth year contains 366 days; determined the obliquity of the ecliptic as 23° 51′ 20″; and measured the length of a degree on the earth's surface, he made this latter about 79 miles, which is too long by nearly 10 miles. Of his work in mathematics we have two extant illustrations: one in a description of an instrument to duplicate a cube; and the other in the rule he gave for preparing a table of prime numbers. The former is given in many books. The latter called "the sieve of Eratosthenes" was as follows: write down all the numbers from 1 upwards; then every second number from 2 is a multiple of 2 and may be cancelled; every third number from 3 is a multiple of 3 and may be cancelled; every fifth number from 5 is a multiple of 5 and may be cancelled; and so on. It has been estimated that it would involve about 300 hours' consecutive work to thus

find the primes in the numbers from 1 to 1,000,000. The labour in determining primes may however be much shortened by observing that if a number can be expressed as the product of two factors one must be less and the other greater than the square root of the number, unless it is a square when the two factors are equal. Hence every composite number must be divisible by a prime which is not greater than its square root.

Eratosthenes lost his sight by ophthalmia, then as now a curse of the valley of the Nile, and refusing to live when he was no longer able to read, he committed suicide by starvation in 194 B.C.

His works exist only in fragments. These were collected and published by G. Bernhardy at Berlin in 1822.

The second century before Christ.

The third century before Christ, which opens with the career of Euclid and closes with the death of Apollonius, is the most brilliant era in the history of Greek mathematics. But the great mathematicians of that century were geometricians, and under their influence attention was directed almost solely to that branch of mathematics. With the methods they used, and to which their successors were by tradition confined, it was hardly possible to make any further great advance: to fill up a few details in a work that was completed in its essential parts was, as Cantor truly remarks, all that could be effected by those methods. It was not till after the lapse of nearly 1800 years that the genius of Descartes opened the way to any further progress in geometry, and I therefore pass over the numerous writers who followed Apollonius with but slight mention. Indeed it may be said roughly that during the next thousand years Pappus was the sole geometrician of great ability; and during this long period almost the only other pure mathematicians of exceptional genius were Hip-

parchus and Ptolemy who laid the foundations of trigonometry, and Diophantus who laid those of algebra.

Early in the second century, circ. 180 B.C., we find the names of three mathematicians who in their own day were very famous.

The first of these was **Hypsicles** who added a fourteenth book to Euclid's *Elements* in which he discussed the regular solids; in another small work he developed the theory of arithmetical progressions, which had been so strangely neglected by the earlier mathematicians.

The second was **Nicomedes** who invented the curve known as the *conchoid* or the mussel-shaped curve. If from a fixed point S a line be drawn cutting a given fixed straight line in Q, and if P be taken on SQ so that the length QP is constant (say d), then the locus of P is the conchoid. Its equation may be put in the form $r = a \sec\theta \pm d$. It is easy with its aid to trisect a given angle or duplicate a cube; and this no doubt was the cause of its invention. Newton made use of this curve in investigating the properties of curves of the third and fourth degree.

The third of these mathematicians was **Diocles** the inventor of the curve known as the *cissoid* or the ivy-shaped curve which like the conchoid was used to give a solution of the duplication problem. He defined it thus: let AOA', BOB' be two fixed diameters of a circle at right angles to one another. Draw two chords QQ' and RR' parallel to BOB' and equidistant from it. Then the locus of the intersection of AR and QQ' will be the cissoid. Its equation can be expressed in the form $y^2(2a - x) = x^3$. Diocles also solved (by the aid of conic sections) a problem which had been proposed by Archimedes, namely to draw a plane which will divide a sphere into two parts whose volumes shall bear to one another a given ratio.

About a quarter of a century later, say about 150 B.C., **Perseus** investigated the various plane sections of the anchor-ring (see p. 42), and **Zenodorus** wrote a treatise on isoperimetrical

figures. Part of the latter work has been preserved; one proposition which will serve to shew the nature of the problems discussed is that "of segments of circles, having equal arcs, the semicircle is the greatest."

Towards the close of this century we find two mathematicians, who by turning their attention to new subjects gave a fresh stimulus to the study of mathematics. These were Hipparchus and Hero.

Hipparchus * was the most eminent of Greek astronomers. He is said to have been born about 160 B.C. at Nicæa in Bithynia; it is probable that he spent some years at Alexandria, but he finally took up his abode at Rhodes where he made most of his observations. Delambre has obtained an ingenious confirmation of the tradition which asserted that Hipparchus lived in the second century before Christ. Hipparchus in one place says that the longitude of a certain star η Canis observed by him was exactly 90°, and it should be noted that he was an extremely careful observer. Now in 1750 it was 116° 4′ 10″, and as the first point of Aries regredes at the rate of 50″·2 a year, the observation was made about 120 B.C.

Except for a short commentary on a poem of Aratus dealing with astronomy all his works are lost, but Ptolemy's great treatise, the *Almagest* (see p. 90), was founded on the observations and writings of Hipparchus, and from the notes there given we infer that the chief discoveries of Hipparchus were as follows. He determined the length of the year within six minutes of its true value. He calculated the inclination of the ecliptic and equator as 23° 51′; it was actually at that time 23° 46′. He estimated the annual precession of the equinoxes as 59″; it is 50·2″. He stated the lunar parallax as 57′, which is nearly correct. He worked out the eccentricity of the solar orbit as 1/24; it is very approximately 1/30. He determined the perigee and mean

* See Delambre, Vol. I. p. 106, &c.

motion both of the sun and moon, and the shifting of the plane of the moon's motion. Finally he obtained the synodic periods of the five planets then known. I leave the details of his observations and calculations to writers who deal specially with astronomy such as Delambre; but it may be fairly said that his work placed the subject for the first time on a scientific basis.

To account for the lunar motion Hipparchus supposed the moon to move with uniform velocity in a circle, the earth occupying a position near (but not at) the centre of this circle. This is equivalent to saying that the orbit is an epicycle of the first order.

This gave the longitude of the moon correct to the first order of small quantities for a few revolutions. To make it correct for any length of time he further supposed that the apse line moved forward about 3° a month, thus giving a correction for evection. He explained the motion of the sun in a similar manner. This theory accounted for all the facts which could be determined with the instruments then in use, and in particular enabled him to calculate the details of eclipses with great accuracy.

He commenced a series of planetary observations to enable his successors to frame a theory to account for their motions; and with great perspicacity he predicted that to do this it would be necessary to introduce epicycles of a higher order; that is, to introduce three or more circles the centre of each successive one moving uniformly on the circumference of the preceding one.

No further advance in the theory of astronomy was made until the time of Copernicus, though the principles laid down by Hipparchus were extended and worked out in detail by Ptolemy.

Investigations such as these naturally led to *trigonometry*, and Hipparchus must be credited with the invention of that subject. It is known that in plane trigonometry he constructed a table of chords of arcs, which is practically the

same as one of natural sines; and that in spherical trigonometry he had some method of solving triangles: but his works are lost, and we can give no details. It is believed however that the elegant theorem printed as Euc. VI. D, and generally known as Ptolemy's Theorem, is due to Hipparchus and was copied from him by Ptolemy. It contains implicitly the addition formulæ for $\sin(A \pm B)$ and $\cos(A \pm B)$; and Carnot shewed how the whole of plane trigonometry could be deduced from it.

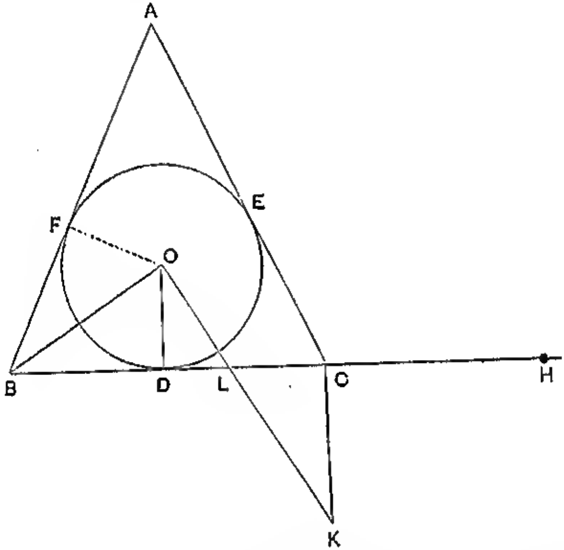
I ought also to add that Hipparchus was the first to indicate the position of a place on the earth by means of its latitude and longitude.

The second of these mathematicians was Hero* of Alexandria (circ. 125 B.C.) who placed engineering and land-surveying on a scientific basis. He was a pupil of Ctesibus, who invented several ingenious machines and is alluded to as if he were a mathematician of note. Hero's principal and most characteristic work is his Μετρικά: this contains (i) some elementary geometry with applications to the determination of the areas of fields of given shapes; (ii) propositions on finding the volumes of certain solids, with applications to theatres, baths, banquet-halls, and so on; (iii) a rule to find the height of an inaccessible object; and (iv) tables of weights and measures. He proved the formula that the area of a triangle is equal to $\frac{1}{2}\{s(s-a)(s-b)(s-c)\}^{\frac{1}{2}}$, where s is the semiperimeter, and a, b, c, the lengths of the sides, and gave as an illustration a triangle whose sides were 13, 14, and 15. He was evidently acquainted with the trigonometry of Hipparchus, but he nowhere quotes a formula, or expressly uses the value of the sine, and it is probable that like the later Greeks he regarded trigonometry as forming an introduction to, and being an integral part of, astronomy.

* See *Recherches sur la vie et les ouvrages d'Héron d'Alexandrie* by T. H. Martin in Vol. IV. of *Mémoires présentés...à l'académie d'inscriptions*, Paris, 1854: *Heronis Alexandrini...reliquiæ*, by F. Hultsch, Berlin, 1864: and an article on the *definitions* of Hero by G. Friedlein in the *Bulletino di Bibliografia*, 1871.

The following is the manner in which he solved the problem to find the area of a triangle ABC the lengths of whose sides are a, b, c.

Let s be the semiperimeter of the triangle. Let the inscribed circle touch the sides in D, E, F, and let O be its



centre. On BC produced take H so that $CH = AF$, therefore $BH = s$. Draw OK at right angles to OB, and CK at right angles to BC. The area ABC or Δ is equal to the sum of the areas OBC, OCA, $OAB = \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr = sr$, i.e. is equal to $BH \cdot OD$. He then shows that the angle $OAF =$ angle CBK; hence the triangles OAF and CBK are similar;

$$\therefore \; BC : CK = AF : OF = CH : OD;$$

$$\therefore \; BC : CH = CK : OD = CL : LD;$$

$$\therefore \; BH : CH = CD : LD;$$

$$\therefore \; BH^2 : CH \cdot BH = CD \cdot BD : LD \cdot BD = CD \cdot BD : OD^2.$$

Hence

$$\Delta = BH \, . \, OD = \{CH \, . \, BH \, . \, CD \, . \, BD\}^{\frac{1}{2}} = \{(s-a) \, s \, (s-c) \, (s-b)\}^{\frac{1}{2}}.$$

I may pass very briefly over Hero's other works. He invented a solution of the duplication problem which is practically the same as that which Apollonius had already discovered (see p. 74). In mechanics he discussed the centre of gravity, the five simple machines, and the problem of moving a given weight with a given power; and here in one place he suggested a way in which the power of a catapult could be tripled. He also wrote on the theory of hydraulic machines. In another treatise he described a theodolite and cyclometer, and pointed out various problems in surveying for which they would be useful. But the most interesting of his smaller works are his Πνευματικά and Αὐτόματα, containing descriptions of about 100 small machines and mechanical toys, many of which are most ingenious. In the former there is an account of a small stationary steam engine which is of the form now known as Avery's patent: it was in common use in Scotland at the beginning of this century, but is not so economical as the form introduced by Watt. There is also an account of a double forcing pump to be used as a fire-engine. It is probable that in the hands of Hero these instruments never got beyond models. It is only recently that general attention has been directed to his discoveries, though Arago had alluded to them in his *éloge* on Watt. An English translation of the Πνευματικά was published by J. G. Greenwood at London in 1851.

An edition of all Hero's works was published by F. Hultsch at Berlin in 1864. Personally I am inclined to doubt whether it is correct to attribute all these to him, but I have followed the opinion of the majority of his recent critics in giving him the credit of them.

All this is very different from the classical geometry and arithmetic of Euclid, or the mechanics of Archimedes. Hero did nothing to extend a knowledge of abstract mathematics; he learnt all that the text-books of the day could teach him, but he was only interested in science on account of its practical

applications, and so long as his results were true he cared nothing for the logical accuracy of the process by which he arrived at them. Thus in finding the area of a triangle he took the square root of the product of four lines. The classical Greek geometricians permitted the use of the square and the cube of a line, because they could be represented geometrically, but a figure of four dimensions is inconceivable, and they would certainly have rejected a proof which involved such a conception.

Several reasons have led modern commentators to think that Hero, who was born in Alexandria, was a native Egyptian. If this be so it affords a curious illustration of the permanence of racial characteristics. The Greeks shewed a special aptitude for geometry and created it; but they never succeeded in reducing arithmetic or algebra to a science except by the aid of geometry. The Semitic races, on the other hand, have produced many eminent algebraists, but not a single geometrician of exceptional ability. Hero spoke and wrote Greek, and was brought up under Greek influence; it is doubtful if he or his contemporaries were aware of the existence of Ahmes' book, and there is no reason to think that they studied the early Egyptian manuscripts; yet the rules he gives, his methods of proof, the figures he draws, the questions he attacks, and even the phrases of which he makes use, recall the earlier work of Ahmes.

The Arabian historians say that Hipparchus and Hero wrote on the solution of a quadratic equation, but this is doubtful.

The first century before Christ.

The successors of Hipparchus and Hero did not avail themselves of the opportunity thus opened of investigating new subjects, but fell back on the well-worn subject of geometry. Amongst the more eminent of these later geometricians were

Theodosius and Dionysodorus, both of whom flourished about 50 B.C.

Theodosius was the author of a complete treatise on the geometry of the sphere. It was edited by Barrow, Cambridge, 1675; and by Nizze, Berlin, 1852.

Dionysodorus is only known to us by his solution of the problem to divide a hemisphere by a plane parallel to its base into two parts, whose volumes shall be in a given ratio. Like the solution by Diocles of the similar problem for a sphere above alluded to (see p. 78), it was effected by the aid of conic sections: it is reproduced in Suter's *Geschichte der mathematischen Wissenschaften*, p. 101. Dionysodorus alone amongst the ancients determined the length of the radius of the earth approximately. He stated it as 42,000 stadia, which, if we take the Olympic stadium of $202\frac{1}{4}$ yards, is little short of 5000 miles. This is not very accurate, but it is nearer the truth than the estimates then generally accepted. We do not know how it was obtained.

The administration of Egypt was definitely undertaken by Rome in 30 B.C. The closing years of the dynasty of the Ptolemies and the earlier years of the Roman occupation of the country were marked by much disorder, civil and political. The studies of the university were naturally interrupted, and it is customary to take this time as the close of the first Alexandrian school.

CHAPTER V.

THE SECOND ALEXANDRIAN SCHOOL. 30 B.C.—641 A.D.

SECTION 1. *The first century after Christ.*
SECTION 2. *The second century after Christ* (Ptolemy).
SECTION 3. *The third century after Christ* (Pappus).
SECTION 4. *The fourth century after Christ* (Diophantus).
SECTION 5. *The Athenian school in the fifth century.*
SECTION 6. *Roman mathematics in the sixth century.*

I CONCLUDED the last chapter by stating that the first school of Alexandria is usually taken as ending at about the same time as the nominal independence of the country. But although the schools at Alexandria suffered from the disturbances which affected the whole Roman world in the transition, in fact if not in name, from a republic to the empire, there was no break of continuity; the teaching in the university was never abandoned; and as soon as order was again established students began once more to flock to Alexandria. This time of confusion was however contemporaneous with a change in the prevalent views of philosophy which thenceforward were mostly neo-platonic or neo-pythagorean, and it therefore fitly marks the commencement of a new period. These mystical opinions re-acted on the mathematical school, and this may partially account for the paucity of good work.

Though Greek influence was still predominant and the Greek language always used, Alexandria now became the in-

tellectual centre for most of the Mediterranean nations which were subject to Rome. It should however be added that the direct connection with it of many of the mathematicians of this time is at least doubtful, but their knowledge was ultimately obtained from the Alexandrian teachers, and they are usually described as of the second Alexandrian school. Such mathematics as were taught at Rome were derived from Greek sources, and we may therefore conveniently leave their consideration for the present.

*The first century after Christ**.

There is no doubt that throughout the first century after Christ geometry continued to be that subject in science to which most attention was devoted. But by this time it was evident that the geometry of Archimedes and Apollonius was not capable of much further extension; and such geometrical treatises as were produced consisted mostly of commentaries on the writings of the great mathematicians of a preceding age. The only original works of any ability were one by Serenus and another by Menelaus. That by Serenus, circ. 70, was on the *plane sections of the cone and cylinder*. This was edited by Halley, Oxford, 1710. That by Menelaus, circ. 98, was on *spherical trigonometry*, investigated in the Euclidean method. This was translated by Halley, Oxford, 1758. The fundamental theorem on which the subject is based is the relation between the six segments of the sides of a spherical triangle, formed by the arc of a great circle which cuts them (book III. prop. 1).

Towards the close of this century, circ. 100, a Jew Nicomachus, who was born at Gerasa in 50 and died circ. 120, published an *Arithmetic*, which (or rather the Latin translation of it) remained for a thousand years a standard

* All dates hereafter given are *anno domini*, unless the contrary is expressly stated.

authority on the subject. Geometrical demonstrations are here abandoned, and the work is a mere classification of the results then known, with numerical illustrations. The evidence for the truth of the propositions enunciated, for I cannot call them proofs, being in general an induction from numerical instances. The object of the book is the study of the properties of numbers, and particularly of their ratios. Nicomachus commences with the usual distinctions between even, odd, prime, and perfect numbers; he next discusses fractions in a tedious and clumsy manner; he then turns to polygonal and to solid numbers; and finally treats of ratio, proportion, and the progressions. The work has been edited by R. Hoche, in Teubner's Library, Leipzig, 1866. Arithmetic of this kind is usually termed Boethian, and the work of Boethius on it was the recognized text-book in the middle ages.

The second century after Christ.

Another arithmetic on very much the same lines as that of Nicomachus was produced by Theon of Smyrna, circ. 130; but it was even less scientific than that of Nicomachus. It has been edited by J. J. de Gelder, Leyden, 1827; and by E. Hiller, Leipzig, 1878.

Another mathematician of about the same date was Thymaridas, who is worthy of notice from the fact that he is the earliest known writer who enunciated an algebraical theorem. He stated that if the sum of any number of quantities be given, and also the sum of every pair which contains one of them, then this quantity will be equal to one $(n-2)$th part of the difference between the sum of these pairs and the first given sum. Thus if

$$x_1 + x_2 + \ldots + x_n = S,$$

and if $\quad x_1 + x_2 = s_2, \; x_1 + x_3 = s_3, \ldots, \text{ and } x_1 + x_n = s_n,$

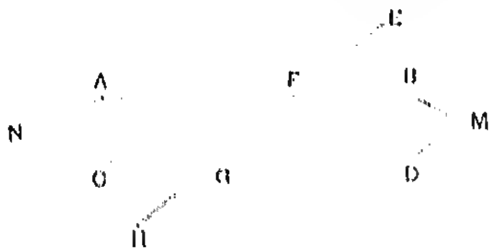
then $\quad x_1 = (s_2 + s_3 + \ldots + s_n - S)/(n-2).$

It does not seem that he used a symbol to denote the unknown

quantity, but he always represented it by the same word, which comes to much the same thing.

About the same time as these writers Ptolemy* of Alexandria, who died in 168, produced his great work on astronomy. This work shows that Ptolemy was a geometrician of the first rank, though it is with the application of geometry to astronomy that he is chiefly concerned. He was the author of numerous other treatises of which all that are extant have been collected in an edition published at Bâle in 1551.

Amongst his writings is a book on *pure geometry* in which he proposed to cancel the twelfth axiom of Euclid on parallel lines, and to prove it in the following manner. Let the straight line $EFGH$ meet the two straight lines AB and



CD so as to make the sum of the angles BFG and FGD equal to two right angles. It is required to prove that AB and CD are parallel. If possible let them not be parallel, then they will meet when produced say at M (or N). But the angle AFG is the supplement of BFG, and is therefore equal to FGD; similarly the angle FGC is equal to the angle BFG. Hence the sum of the angles AFG and FGC is equal to two right angles, and the lines BA and DC will therefore if produced meet at N (or M). But two straight lines cannot enclose a space, therefore AB and CD cannot meet when produced, that is they are parallel. Conversely, if AB and CD be

* See the Article *Ptolemaeus Claudius* by A. de Morgan in *Smith's Dictionary of Greek and Roman Biography*. London, 1849.

parallel, then AF and CG are not less parallel than FB and GD; and therefore whatever be the sum of the angles AFG and FGC such must also be the sum of the angles FGD and BFG. But the sum of the four angles is equal to four right angles, and therefore the sum of the angles BFG and FGD must be equal to two right angles.

Ptolemy wrote another work to shew that there could not be more than three dimensions in space: and he also discussed *orthographic and stereographic projections* with special reference to the construction of sun-dials. A book on *optics* is sometimes attributed to him, but it is most probably not genuine.

But the work on which the reputation of Ptolemy rests is his astronomy, usually called the *Almagest*. The name is derived from the Arabic title *al midschisti*, which is said to be a corruption of *μεγίστη* [*μαθηματικὴ*] *σύνταξις*. This book is a splendid testimony to the ability of the author. It is founded on the writings of Hipparchus, and though it did not sensibly advance the theory of the subject, it presents the views of the older writer with a completeness and elegance which will always make it a standard treatise. We gather from it that Ptolemy made observations at Alexandria from the years 125 to 150; he however was but an indifferent practical astronomer, and the observations of Hipparchus are generally more accurate than those of his expounder.

The work is divided into thirteen books. In the first book Ptolemy discusses various preliminary matters; treats of trigonometry plane and spherical; gives a table of chords, i.e. of natural sines (which is probably taken from the lost work of Hipparchus, but which is very accurate), and explains the obliquity of the ecliptic. The second book is chiefly devoted to phenomena depending on the spherical form of the earth: he remarks that the explanations would be much simplified if the earth were supposed to rotate on its axis once a day, but points out that this hypothesis is inconsistent with known facts. In the third book he explains the motion of the sun round the earth by means of excentrics and epicycles

and in books four and five he treats the motion of the moon in a similar way. The sixth book is devoted to the theory of eclipses. The seventh and eighth books contain a catalogue of the fixed stars (probably copied from Hipparchus): and in another work he added a list of annual sidereal phenomena. The remaining books are given up to the theory of the planets.

It should be added that the *Almagest* became at once the standard work on astronomy and remained so till Copernicus and Kepler shewed that the sun and not the earth must be regarded as the centre of the solar system.

The idea of eccentrics and epicycles on which the theories of Hipparchus and Ptolemy are based has been often ridiculed in modern times. No doubt at a later time, when more accurate observations had been made, the necessity of introducing epicycle on epicycle in order to bring the theory into accordance with the facts made it very complicated. But de Morgan has acutely observed that in so far as the ancient astronomers supposed that it was necessary to resolve every celestial motion into a series of uniform circular motions they erred greatly, but that if the hypothesis be regarded as a convenient way of expressing known facts it is not only legitimate, but convenient. It was as good a theory as with their instruments and knowledge it was possible to frame, and in fact it exactly corresponds to the expression of a given function as a sum of sines or cosines, a method which is of frequent use in modern analysis.

In spite of the trouble taken by Delambre it is almost impossible to separate the results due to Hipparchus from those due to Ptolemy. But Delambre and de Morgan agree in thinking that the observations quoted, the fundamental ideas, and the explanation of the apparent solar motion are wholly due to Hipparchus; while all the lunar and planetary theories, beyond the general idea of them, is due to Ptolemy.

The third century after Christ.

Ptolemy had not only shown that geometry could be applied to astronomy, but had indicated how new methods of analysis like trigonometry might be thence developed. He found however no successors to take up the work he had commenced so brilliantly; and we must look forward 150 years before we find another geometrician of any eminence. That geometrician was Pappus who lived and taught at Alexandria about the end of the third century. We know that he had numerous pupils, and it is probable that he temporarily revived an interest in the study of geometry.

He wrote several books, but the only one which has come down to us is his Συναγωγή, a collection of mathematical papers arranged in eight books of which the first and part of the second have been lost. This collection was intended to be a synopsis of Greek mathematics together with comments and additional propositions by the editor. A careful comparison of various extant works with the account given of them in this book shows that it is trustworthy: and we rely on it for most of our knowledge of the works now lost. It is not arranged chronologically, but all the treatises on the same subject are grouped together; and it is most probable that it gives roughly the order in which the classical authors were read at Alexandria. It has been edited by F. Hultsch, Berlin, 1876—8.

The first five books deal with geometry exclusive of conic sections: the sixth with astronomy including, as subsidiary subjects, optics and trigonometry: the seventh with analysis, conics, and porisms: and the eighth with mechanics. The last two books contain a good deal of original work by Pappus; at the same time it should be remarked that in two or three cases he has been detected in appropriating proofs from earlier authors, and it is possible he may have done this in other cases.

Subject to this suspicion we may say that he discovered the focus in the parabola, and the directrix in the conic

equations. In geometry he propounded and solved the problem to inscribe in a given circle a triangle whose sides produced shall pass through three collinear points*. In mechanics, he showed that the centre of mass of a triangle is the same as that of an inscribed triangle whose vertices divide each of the sides of the original triangle in the same ratio; he also discovered the two theorems on the surface and volume of a solid of revolution† which are still quoted in text-books under his name. These are only samples of many brilliant but isolated theorems which are here enunciated.

His work as a whole and his comments shew that he was a geometrician of great power; but it was his misfortune to live at a time when but little interest was taken in geometry, and when the subject had been practically exhausted.

A small tract on multiplication and division of sexagesimal fractions is possibly due to Pappus, and was certainly written about this time. It was edited by C. Henry, Halle, 1879 and is valuable as an illustration of practical Greek arithmetic.

The fourth century after Christ.

Throughout the second and third centuries, that is from the time of Nicomachus, interest in geometry had steadily decreased, and more and more attention had been paid to the

* This problem (book vii, prop. 107) was in the eighteenth century generalised by Cramer by supposing the three given points to be anywhere; and was celebrated for its difficulty. It was sent in 1742 as a challenge to Castillon, who solved it in 1776 (*Mém. de l'Acad. de Berlin*). Lagrange, Euler, Lhuilier, Fuss, and Lexell also gave solutions in 1780. A few years later the problem was set to a Neapolitan lad Ottaiano who was only 16, but who had shewn great mathematical ability, and he extended it to the case of a polygon of n sides which have to pass through n given points, and gave a very simple and elegant solution. Poncelet has recently extended it to conics of any species and subject to other restrictions.

† The volume is equal to the product of the area of the curve, and the length of the path described by its centre of mass. The surface is equal to the product of the perimeter of the curve and the length of the path described by its centre of mass.

theory of numbers, though the results were in no way commensurate with the time devoted to the subject. It will be remembered that Euclid used lines as symbols for any magnitudes, and investigated a number of theorems about numbers in a strictly scientific manner, but he confined himself to cases where a geometrical representation was possible. There are indications in the works of Archimedes that he was prepared to carry the subject much further: he introduced numbers into his geometrical discussions (see p. 64), and divided lines by lines, but he was fully occupied by other researches, and had no time to devote to arithmetic. Hero abandoned the geometrical representation of numbers, but he, Nicomachus, and other later writers on arithmetic did not succeed in creating any other symbolism for numbers in general, and thus when they enunciated a theorem they were content to verify it by a large number of numerical examples. They probably knew how to solve a quadratic equation with numerical coefficients; for, as Montucla pointed out, the propositions Euc. VI. 28 and VI. 29 give geometrical solutions of the equations $x^2 + px = q$ and $x^2 - px = q$, but if so this represented their highest attainment.

It would seem, then, that in spite of the time given to its study, arithmetic and algebra had not made any sensible advance since the time of Archimedes. It is however interesting to notice that the problems which excited most interest in the third century are those which would naturally lead to simple equations. They may be illustrated from a collection of questions which was made by Metrodorus at the beginning of the next century, about 310. Some of them are due to the editor, but some are of an anterior date, and they all fairly illustrate the way in which arithmetic was leading up to algebraical methods. The following are typical examples. "Four pipes discharge into a cistern: one fills it in one day; another in two days; the third in three days; the fourth in four days; if all run together how soon will they fill the cistern?" "Demochares has lived a fourth of his life

as a boy; a fifth as a youth; a third as a man; and has spent thirteen years in his dotage: how old is he?" "Make a crown of gold, copper, tin, and iron weighing 60 minae: gold and copper shall be two-thirds of it; gold and tin three-fourths of it; and gold and iron three-fifths of it: find the weights of the gold, copper, tin, and iron which are required." The last is included in Thymaridas's theorem quoted on p. 88.

The German commentators have pointed out that though these problems lead to simple equations, they can all be solved by geometrical methods, the unknown quantity being represented by a line. Dean Peacock has also remarked that they can be solved by the method used in similar cases by the Arabians and many medieval writers. This method, usually known as *the rule of false assumption*, consists in assuming any number for the unknown quantity, and if on trial it does not satisfy the given conditions, correcting it by a simple proportion as in rule of three. For example, in the second problem, suppose we assume Demochares' age to be 40, then, by the given conditions, he would have spent $8\frac{2}{3}$ (and not 13) years in his dotage, and therefore we have the proportion

$$8\tfrac{2}{3} : 13 = 40 : \text{his actual age},$$

hence his actual age is 60.

But the most recent writers on the subject think that the problems were solved by *rhetorical algebra*, that is by a process of algebraical reasoning expressed in words, and without the use of any symbols. This according to Nesselmann is the first stage in the development of algebra, and we find it used both by Ahmes and by the earliest Arabian, Persian, and Italian algebraists.

On this view then a rhetorical algebra had been gradually evolved by the Greeks. Its development was however very imperfect. Hankel, who is no unfriendly critic, says that the results attained as the net outcome of 600 years' work on the theory of numbers are, whether we look at the form or the substance, unimportant or even childish, and are not in any way the commencement of a science. In the midst of this

decaying interest in geometry, and these feeble attempts at algebraic-arithmetic, a single algebraist of marked originality suddenly appeared who created what was practically a new science. This was Diophantus who was the first* mathematician to introduce a system of abbreviations for those operations and quantities which constantly recur, though in using them he observed all the rules of grammatical syntax. The resulting science is called by Nesselmann *syncopated algebra*: it is a sort of shorthand. Broadly speaking, it may be said that European algebra did not advance beyond this stage until the close of the sixteenth century.

Modern algebra has progressed one stage further and is entirely *symbolic*; that is, it has a language of its own, and a system of notation which has no obvious connection with the things represented, while the operations are performed according to certain rules which are independent of and distinct from the laws of grammatical construction.

All that we know of Diophantus† is that he lived at Alexandria, and that most likely he was not a Greek. Even the date of his career is uncertain, but it may be put down as probably about the beginning or middle of the fourth century, that is shortly after the death of Pappus. He was 84 when he died.

Diophantus not only invented a new language in algebra for expressing results that were already known, but applied the subject to a number of problems which had previously baffled all investigation. His work however excited no interest among

* Cantor (p. 385) thinks there are traces of the use of algebraic symbolism in Pappus. Friedlein (p. 19) mentions a Greek papyrus in which the signs / and Ↄ are used for addition and subtraction respectively; but the date of the MS. is not mentioned. De Morgan thinks that Diophantus only systematized the knowledge which was familiar to his contemporaries. The question of the originality of Diophantus is a very difficult one, but the opinion expressed in the text is that of most of the recent commentators.

† See *Diophantos of Alexandria* by T. L. Heath, Cambridge, 1885.

his contemporaries, and with his death his writings and methods fell into an oblivion from which they were not rescued for many centuries. The first translation or edition of his *Arithmetic* which was published in Europe was by Xylander in 1575, but the book was known to the Arabs at an earlier date.

Diophantus wrote a short essay on polygonal numbers; a treatise on algebra, which has come down to us in a mutilated condition; and a work on porisms, which is lost.

The *Polygonal numbers* was probably his earliest work. In this he abandoned the empirical method of Nicomachus, and reverted to the old and classical system by which numbers are represented by lines, a construction is (if necessary) made, and a strictly deductive proof follows: it may be noticed that in it he quotes propositions such as Euc. II. 3 and II. 8, as referring to numbers and not to any magnitudes.

His chief work is his *Arithmetic*. This is really a treatise on algebra; algebraic symbols are used, and the problems are treated analytically. It will be observed that nearly all modern algebra is analytical, and tacitly assumes that the steps are reversible. I propose to consider successively the notation, the methods of analysis employed, and the subject-matter of this work.

First as to the notation. Diophantus always employed a symbol to represent the unknown quantity in his equations, but as he had only one symbol he could never use more than one unknown at a time (see p. 101). The unknown quantity is called ὁ ἀριθμός, and is represented by ς′ or ς°′. It is usually printed as s. In the plural it is denoted by ss or ssοἰ. This symbol may be a corruption of αρ, or possibly is an old hieratic symbol for the word heap (see p. 9), or it may stand for the final sigma of the word. The square of the unknown is called δύναμις, and denoted by δυ: the cube κύβος, and denoted by κυ; and so on up to the sixth power.

The coefficients of the unknown quantity are numbers, and they are written immediately after the quantity they multiply.

Thus $\varsigma'\bar{\alpha} \equiv x$, $\varsigma\varsigma^{o\hat{\iota}}\,\overline{\iota\alpha} \equiv \overline{\varsigma\varsigma}\,\overline{\iota\alpha} \equiv 11x$. An absolute term is regarded as a certain number of units or $\mu o\nu\acute{\alpha}\delta\epsilon\varsigma$, which are represented by $\mu^{\hat{o}}$: thus $\mu^{\hat{o}}\bar{\alpha} \equiv 1$, $\mu^{\hat{o}}\,\overline{\iota\alpha} \equiv 11$.

There is no sign for addition beyond mere juxtaposition. Subtraction is represented by $\pitchfork$, and this symbol affects all the symbols that follow it. Equality is represented by ι. Thus

$$\kappa^{\hat{\upsilon}}\bar{\alpha}\;\overline{\varsigma\varsigma}\,\bar{\eta}\;\pitchfork\;\delta^{\hat{\upsilon}}\bar{\epsilon}\;\mu^{\hat{o}}\bar{\alpha}\;\iota\;\varsigma\bar{\alpha}$$

represents $$(x^3 + 8x) - (5x^2 + 1) = x.$$

Diophantus also introduced a somewhat similar notation for fractions involving the unknown quantity, into the details of which I need not here enter.

It will be noticed that all these symbols are mere abbreviations for words: and he reasoned out his proofs writing these abbreviations in the middle of his text. In most manuscripts there is a marginal summary in which the symbols alone are used, and which is really symbolic algebra; but this is probably the addition of some scribe of later times.

This introduction of a contraction or a symbol instead of a word to represent an unknown quantity marks a greater advance than anyone not acquainted with the subject would imagine: and those who have never had the aid of some such abbreviated symbolism find it almost impossible to understand complicated algebraical processes. It is likely enough that it might have been introduced earlier, but for the unlucky system of numeration adopted by the Greeks, by which they used all the letters of the alphabet to denote particular numbers, and thus made it impossible to employ them to represent any number.

Next as to the knowledge of algebraic methods shown in the book. Diophantus commences with some definitions which include an explanation of his notation, and in giving the symbol for *minus*, he states that a subtraction multiplied by a subtraction gives an addition; by this he means that the product of $-b$ and $-d$ in the expansion of $(a-b)(c-d)$ is $+bd$: but in applying it, he always takes care that the

numbers a, b, c, d are so chosen that a is greater than b and c is greater than d.

The whole of the work itself, or at least as much as is now extant, is devoted to solving problems which lead to equations. It contains the rule for solving a simple equation of the first degree, and a binomial quadratic. The rule for solving quadratic equations generally is in one of the lost books, but where the equation was of the form $ax^2+bx+c=0$ he seems to have multiplied by a, and then "completed the square" in much the same way as is now done; if the roots of x are negative or irrational the equation is rejected as "impossible," and even when both roots are positive he never gives more than one, always taking the positive value of the square root. Diophantus solves one cubic equation, namely $x^3+x=4x^2+4$ (book VI., prob. 19).

The greater part of the work is however given up to indeterminate equations between two or three variables. When the equation is between two variables then, if it is of the first degree, he assumes a suitable value for one variable and solves the equation for the other. Most of his equations are of the form $y^2=Ax^2+Bx+C$. Whenever A or C is absent he is able to solve the equation completely. When this is not the case, then if $A=a^2$ he assumes $y=ax+m$; if $C=c^2$ he assumes $y=mx+c$; and lastly if the equation can be put in the form $y^2=(ax\pm b)^2+c^2$ he assumes $y=mx$: where in each case m has some particular numerical value suitable to the problem under consideration. A few particular equations of a higher order occur, but in these he generally alters the problem so as to enable him to reduce the equation to one of the above forms.

The indeterminate equations involving three variables, or "double equations" as he calls them, which he considers are of the form

$$\left.\begin{array}{l} y^2=Ax^2+Bx+C \\ z^2=ax^2+bx+c \end{array}\right\}.$$

If A and a both vanish he solves them in one of two ways.

It will be enough to give one of his methods which is as follows: he subtracts and thus gets an equation of the form $y^2 - z^2 = mx + n$; hence if $y \pm z = \lambda$, then $y \mp z = (mx + n)/\lambda$; and solving he finds y and z. His treatment of "double equations" of a higher order lacks generality and depends on the particular numerical conditions of the problem.

Lastly, as to the matter of the book. The problems he attacks, and the analysis he uses, are so various that they cannot be concisely described and I have therefore selected six typical problems to illustrate his methods. What seems to strike his critics most is the ingenuity with which he selects as his unknown some quantity which leads to equations such as he can solve, and the artifices by which he finds numerical solutions of his equations.

I select the following as characteristic examples:

(i) *Find four numbers, the sum of every arrangement three at a time being given; say, 22, 24, 27, and 20* (book I., prob. 17). Let x be the sum of all four numbers; $\therefore$ the numbers are

$$x - 22, \quad x - 24, \quad x - 27, \text{ and } x - 20;$$

$$\therefore \quad x = (x - 22) + (x - 24) + (x - 27) + (x - 20);$$

$$\therefore \quad x = 31;$$

$\therefore$ the numbers are 9, 7, 4, and 11.

(ii) *Divide a number, such as 13 which is the sum of two squares, 4 and 9, into two other squares* (book II., prob. 10). He says that since the given squares are 2^2 and 3^2 he will take $(x + 2)^2$ and $(mx - 3)^2$ as the required squares, and will assume $m = 2$;

$$\therefore \quad (x + 2)^2 + (2x - 3)^2 = 13;$$

$$\therefore \quad x = 8/5;$$

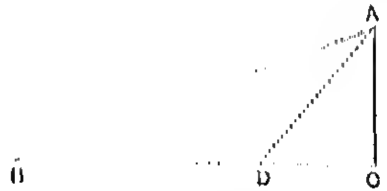
$\therefore$ the required squares are 324/25 and 1/25.

(iii) *Find two squares such that the sum of the product and either is a square* (book II., prob. 29). Let x^2 and y^2 be the numbers. Then $x^2y^2 + y^2$ and $x^2y^2 + x^2$ are squares. The first will be a square if $x^2 + 1$ be a square; which he therefore

assumes may be taken equal to $(x-2)^2$, hence $x=3/4$. He has now to make $9(y^2+1)/16$ a square; to do this he assumes that $9y^2+9=(3y-4)^2$, hence $y=7/24$. Therefore the squares required are $9/16$ and $49/576$.

It will be recollected that Diophantus had only one symbol for an unknown quantity; and in this example he begins by calling the unknowns x^2 and 1, but as soon as he has found x, he then replaces the 1 by the symbol for the unknown quantity, and finds it in its turn.

(iv) *To find a [rational] right-angled triangle such that the line bisecting an acute angle is rational* (book vi., prob. 18). His solution is as follows. Let the bisector $AD=5x$, and let



$DC=3x$, hence $AC=4x$. Next let BC be a multiple of 3, say 3; $\therefore BD=3-3x$; hence $AB=4-4x$ (by Euc. vi. 3). Hence $(4-4x)^2=3^2+(4x)^2$ (Euc. i. 47); $\therefore x=7/32$. Multiplying by 32 we get for the sides of the triangle 28, 96, and 100; and for the bisector 35.

(v) *A man buys x measures of wine, some at 8 drachmae a measure, the rest at 5. He pays for them a square number of drachmae, such that if 60 be added to it, the resulting number is x^2. Find the number he bought at each price* (book v., prob. 33).

The price paid was x^2-60, hence $8x>x^2-60$ and $5x<x^2-60$. From this it follows that x must be greater than 11 and less than 12.

Again x^2-60 is to be a square; suppose it is equal to $(x-m)^2$, then $x=(m^2+60)/2m$, we have therefore

$$11 < \frac{m^2 + 60}{2m} < 12;$$

$$\therefore \quad 19 < m < 21.$$

Diophantus therefore assumes that m is equal to 20, which gives him $x = 11\frac{1}{2}$; and makes the total cost, i.e. $x^2 - 60$, equal to $72\frac{1}{4}$ drachmae.

He has next to divide this cost into two parts which shall give the cost of the 8 drachmae measures and the 5 drachmae measures respectively. Let these parts be y and z.

Then $$\tfrac{1}{5}z + \tfrac{1}{8}(72\tfrac{1}{4} - z) = 11\tfrac{1}{2}.$$

Therefore $$z = \frac{5 \times 79}{12}, \text{ and } y = \frac{8 \times 59}{12}.$$

Therefore the number of 5 drachmae measures was 79/12, and of 8 drachmae measures was 59/12.

From the enunciation of this problem it would seem that the wine was of a poor quality, and M. Tannery has ingeniously suggested that the prices mentioned for such a wine are higher than were usual until after the year 200 A.D. He therefore rejects the view which was formerly held that Diophantus lived in the second century of our era. De Morgan had however previously shown that this opinion was certainly wrong. M. Tannery inclines to put Diophantus half a century earlier than I have supposed.

I mentioned that Diophantus wrote a third work entitled

Diophantus exercised no perceptible influence on Greek mathematics, but his *Arithmetic* when translated into Arabic in the tenth century influenced the Arabian school, and so indirectly affected the progress of European mathematics. A copy of the work was discovered in 1462; it was translated into Latin and published in 1575: the translation excited general interest, but by that time the European algebraists had already advanced beyond the point at which Diophantus had left off.

The names of two commentators will practically conclude the long roll of Alexandrian mathematicians. The first of these is **Theon** of Alexandria who flourished about 370. He was not a mathematician of any originality, but we are indebted to him for an edition of Euclid's *Elements*, and a commentary on the *Almagest*; the latter gives a great deal of miscellaneous information about the numerical methods used by the Greeks. The other was **Hypatia** the daughter of Theon. She was more distinguished than her father, and was the last Alexandrian mathematician of any general reputation; she wrote a commentary on the *Conics* of Apollonius, and possibly some other works. She was murdered at the instigation of the Christians in 415.

The fate of Hypatia may serve to remind us that the Christians, as soon as they became the dominant party in the state, showed themselves bitterly hostile to all forms of learning. That very singleness of purpose which had at first so materially aided their progress developed into a one-sidedness which refused to see any good outside their own body; those who did not actively assist them were persecuted, and the manner in which they carried on their war against the old schools of learning is faithfully pictured in the pages of Kingsley's novel. The final establishment of Christianity in the East marks the end of the Greek scientific schools, though they nominally continued to exist for two hundred years more.

The Athenian school (in the fifth century).

The hostility of the Eastern church to Greek science is further illustrated by the fall of the later Athenian school. This school occupies but a small space in our history. Ever since Plato's time a certain number of professional mathematicians had lived at Athens; and about 420 this school again acquired considerable reputation, largely in consequence of the numerous students who after the murder of Hypatia migrated there from Alexandria. The most celebrated of its pupils were Proclus, Damascius, and Eutocius.

Proclus* was born at Constantinople in February 412 and died at Athens on April 17, 485. He wrote a commentary on Euclid's *Elements*, of which that part which deals with the first book is extant, and contains a great deal of valuable information on the history of Greek mathematics: he is verbose and dull, but luckily he has preserved for us quotations from other and better authorities. His commentary has been recently edited by G. Friedlein, Leipzig, 1867.

Damascius of Damascus, circ. 490, was educated at Athens and subsequently lectured there. He added to Euclid's *Elements* a 15th book on the inscription of one regular solid in another.

Eutocius, circ. 510, wrote commentaries on the first four books of the *Conics* of Apollonius and on several works of Archimedes. His works have never been edited, though they would seem to well deserve it.

This later Athenian school was carried on under great difficulties owing to the opposition of the Christians. Proclus, for example, was repeatedly threatened with death because he was "a philosopher." His remark "After all what does my body matter? it is the spirit that I shall take with me when I die," which he made to some students who had offered to defend him has often been quoted. The Christians after seve-

* See *Untersuchungen über die neu aufgefundenen Scholien des Proklus* by Knoche, Herford, 1865.

and ineffectual attempts at last got a decree from Justinian in 529 that "heathen learning" should no longer be studied at Athens.

The church at Alexandria was less influential, and the city was more remote from the centre of civil power. The schools there were thus suffered to continue, though their existence was of a very precarious character. Under these conditions mathematics continued to be read for another hundred years, but all interest in the study had gone.

Roman mathematics in the sixth century.

I ought not to conclude this part of the history without any mention of Roman mathematics, for it was through Rome that mathematics first passed into the curriculum of mediæval Europe, and in Rome all modern history has its origin. There is however very little to say on the subject. The chief study of the place was in fact the art of government, whether by law, by persuasion, or by those material means on which all government ultimately rests. There were no doubt professors who could teach the results of Greek science but there was no demand for a school of mathematics. Italians who wished to learn more than the elements of the science went to Alexandria, or to places which drew their inspiration from Alexandria.

The subject as taught in the mathematical schools at Rome* seems to have been confined in arithmetic to the art of calculation (no doubt by the aid of the abacus) and perhaps some of the easier parts of the work of Nicomachus; and in geometry to a few practical rules: though some of the arts founded on a knowledge of mathematics (especially that of surveying) were carried to a high pitch of excellence. It would seem also that special attention was paid to the representation of numbers by

* See *Die Römischen Agrimensoren* by M. Cantor, Leipzig, 1875. See also *Matériaux pour servir à l'histoire comparée des sciences mathématiques chez les Grecs et les Romains* by L. A. Sédillot, Paris, 1845—49.

signs. The manner of indicating numbers up to ten by the use of fingers must have been in practice from quite early times; but about the first century it had been developed by the Romans into a finger-symbolism by which numbers up to 10,000 or perhaps more could be represented: this would seem to have been taught in the Roman schools. The system would hardly be worth notice but that its use has still survived in the Persian bazaars.

I am not aware of any Latin work on the principles of mechanics; but there were numerous books on the practical side of the subject which dealt elaborately with architectural and engineering problems. We may judge what they were like by the *Mathematici Veteres*, which is a collection of various short treatises on catapults, engines of war, &c. (an edition was published in Paris, in 1693): and by the Κεστοί, written in 200, which contains, amongst other things, rules for finding the breadth of a river when the opposite bank is occupied by an enemy, how to signal with a semaphore, &c.

In the sixth century Boethius published a geometry and an arithmetic. The former contains a few propositions from Euclid, the latter is a free translation from Nicomachus. There is nothing original in either work, but they formed the standard text-books of western Europe for some six or seven centuries, and thus it is necessary to consider them, though their intrinsic value is very slight.

Boethius, or as the name is sometimes written *Boetius*, born at Rome about 475 and died in 526, belonged to a family which for the two preceding centuries had been esteemed the most illustrious in Rome. It was formerly believed that he was educated at Athens, but though this is somewhat doubtful he was exceptionally well read in Greek literature and science. He would seem to have wished to devote his life to literary pursuits; but recognizing "that the world would only be happy, either when kings became philosophers, or philosophers kings," he yielded to the pressure put on him and took an active share in politics. He was cele-

'ated for his extensive charities, and what in those days was ıry rare the care that he took to see that the recipients were ɔrthy of them. He was elected consul at an unusually early e, and took advantage of his position to reform the coinage ıd to introduce the public use of sun-dials, water-clocks, &c. e reached the height of his prosperity in 522, when his two ns were inaugurated as consuls. His integrity and attempts protect the provincials from the plunder of the public officials ought on him the hatred of the Court. He was sentenced death while absent from Rome, seized at Ticinum, and in e baptistery of the church there tortured by drawing a cord und his head till the eyes were forced out of the sockets, d finally beaten to death with clubs on Oct. 23, 526. Such least is the account that has come down to us. At a later ne his merits were recognized, and tombs and statues erected his honour by the state.

Boethius was the last Roman of any note who studied the ıguage and literature of Greece, and his works afforded to ɔdiæval Europe the only means of entering into the in- llectual life of the old world. His importance in the history literature is thus very great, but it arises merely from the ɔident of the time at which he lived. After the introduction Aristotle's works in the thirteenth century his fame died 'ay, and he has now sunk into an obscurity which is as great was once his reputation. He is best known by his *Conso- io*, which was translated by Alfred the Great into Anglo- xon. For our purpose it is sufficient to note that early ɔdiæval mathematics was entirely founded on his geometry d arithmetic.

His *Geometry* consists of the enunciations (only) of the first ɔk of Euclid, and of a few selected propositions in the third d fourth books, but with numerous practical applications to ding areas, &c. He adds an appendix with proofs of the st three propositions to shew that the enunciations may be ied on. He also wrote an *Arithmetic*, founded on that of comachus. These works have been edited by G. Friedlein,

the empire tending to divert men's thoughts into other channels. In 632 Mohammed died, and within ten years his successors had subdued Syria, Palestine, Mesopotamis, Persia, and Egypt. The precise date on which Alexandria fell is doubtful, but the most reliable Arab historians give Dec. 10, 641—a date which at any rate is correct within eighteen months.

With the fall of Alexandria the long history of Greek mathematics came to a conclusion. It seems probable that the greater part of the famous university library and museum had been destroyed by the Christians a hundred or two hundred years previously, and what remained was unvalued and neglected. Some two or three years after the first capture of Alexandria a serious revolt occurred in Egypt which was ultimately put down with great severity. I see no reason to doubt the truth of the account that after the capture of the city the Mohammedans destroyed such university buildings and collections as were still left. It is said that when the Arab commander ordered the library to be burnt, the Greeks made such energetic protests that he consented to refer the matter to the caliph Omar. The caliph returned the answer, "As to the books you have mentioned, if they contain what is agreeable with the book of God, the book of God is sufficient without them; and if they contain what is contrary to the book of God, there is no need for them; so give orders for their destruction." The account goes on to say that they were burnt in the public baths of the city, and that it took six months to consume them all.

CHAPTER VI.

THE BYZANTINE SCHOOL. 641—1543.

It will be convenient to consider the Byzantine sch connection with the history of Greek mathematics. Afte capture of Alexandria by the Mahommedans the major the philosophers, who had previously been teaching migrated to Constantinople which then became the cen Greek learning in the East and remained so for 900 But though its history covers such an immense interv time it is utterly barren of any scientific interest; a chief merit is that it preserved for us the works of the ent Greek schools. The revelation of these works t West in the fifteenth century was one of the most imp sources of the stream of modern European thought, an history of the school may be summed up by saying t played the part of a conduit pipe in conveying to us the r of an earlier and brighter age.

The time was one of constant war, and men's minds d the short intervals of peace were wholly occupied with logical subtleties and pedantic scholarship. I should not mentioned any of the following writers had they lived i Alexandrian period, but in default of any others they m noticed, as illustrating the character of the school.

One of the earliest members of the Byzantine scho **Hero of Constantinople**, circ. 900, sometimes called the yo to distinguish him from Hero of Alexandria (see p. 81). is great difficulty in separating the works of these two w and some think that the expression for the area of a tr given on page 81 is due to Hero the younger; it cer

possesses the characteristics of the work of this time. Hero would seem to have written on geodesy and mechanics as applied to engines of war.

During the tenth century two emperors Leo VI. and Constantine VII. shewed considerable interest in astronomy and mathematics, but the stimulus thus given to their study was only temporary.

In the eleventh century **Psellus**, born in 1020, wrote a pamphlet on the quadrivium. It is now in the National Library at Paris, MS. No. 2338; it was printed at Bâle in 1556. He also wrote a *Compendium mathematicum*, which was printed at Leyden in 1647.

In the fourteenth century we find the names of three monks who paid attention to mathematics.

The first of the three was **Planudes**; he wrote a commentary on the two first books of Diophantus, published by Xylander, Bâle, 1575; a work on Hindoo arithmetic published by C. J. Gerhardt, Halle, 1865; and another on proportions which is now in the National Library at Paris, MS. No. 2358.

The next was a Calabrian monk named **Barlaam**, who died in 1348. He was the author of a work on the Greek methods of calculation, from which we derive a good deal of our information as to the way in which the Greeks practically treated fractions; this was published in Paris in 1606. Barlaam seems to have been a man of great intelligence. He was sent as an ambassador to the pope at Avignon, and acquitted himself of a difficult mission very creditably; while there he taught Greek to Petrarch. He was famous at Constantinople for the ridicule he threw on the preposterous pretensions of the monks at Mount Athos (see Gibbon's *Decline*, VI., pp. 129, 346) who taught that those who joined them could by standing naked, resting their beards on their breasts, and steadily regarding their stomachs see a mystic light which was the essence of God. Barlaam advised them to substitute the light of reason for that of their stomachs—a piece of advice which nearly cost him his life.

The last of these monks was **Argyrus**, who died in 1372. He wrote three astronomical tracts the manuscripts of which are in the libraries at the Vatican, Leydon, and Vienna: one on geodesy the manuscript of which is at the Escurial: one on geometry the manuscript of which is in Paris, MS. No. 2418: and one on trigonometry the manuscript of which is at the Bodleian, Oxford.

In the fourteenth or perhaps the fifteenth century Nicholas of Smyrna wrote an arithmetic which is now in the National Library at Paris, MS. No. 2428. He also wrote an account of the finger-symbolism (see p. 106) which the Romans had introduced into the East and was then current there; the latter is described by Bede and would therefore seem to have been known as far west as Britain: Jerome also alludes to it.

In the fifteenth century **Pachymeres** wrote tracts on arithmetic, geometry, and four mechanical machines. A few years later **Moschopulus** wrote a treatise on magic squares*, the formation of which was then a favourite amusement as they were supposed to possess mystical properties: and in particular

* A *magic square* of the n^{th} order is formed by arranging the first n^2 natural numbers in the form of a square so that the sum of the numbers in every row, every column, and in the two diagonals shall be the same and equal to $\frac{1}{2}n(n^2+1)$. For example, the following are magic squares of the 4^{th} and 5^{th} orders:

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

A magic square of an odd order can always be written down at once (though I do not know any proof of the rule usually given). A magic square of the order $4m$ can also be written down: but a square of the order $4m+2$ can only be found empirically. If magic squares of the orders p and q are known, it is possible to form one of the order pq.

when engraved on silver plates to be a charm against the plague. One is figured in the picture of melancholy painted about the year 1500 by Albert Dürer. In later times these squares have formed a favorite subject for many writers. Euler in particular seems to have been fascinated by it, and in a memoir published in the *Hist. de l'Acad. des Sciences, Berlin*, 1759, gives magic squares of various orders which are subject to additional restrictions; such as that the sum of pairs of numbers opposite to and equidistant from the middle figure of a magic square of an odd order may be the same. Some of the more recent articles on the subject will be found in the *Quarterly Journal of Pure and Applied Mathematics*, Vol. x., p. 186; Vol. xi., pp. 57, 123, 213; Vol. xii., p. 213: *The Messenger of Mathematics*, Vol. ii.: the *Nouv. Corr. Math.* Vol. ii., pp. 161, 193; and the *Report* for 1880 of the *French association for the advancement of science.* Moschopulus was the earliest writer who attempted to deal with the mathematical theory of such squares. His work exists in manuscript in the National Library at Paris (ms. No. 2428). Moschopulus died in Italy circ. 1470.

Constantinople was captured by the Turks in 1453, and the last semblance of a Greek school of mathematics then disappeared. Numerous Greeks took refuge in Italy. In the West the memory of Greek science had vanished, and even the names of all but a few Greek writers were unknown; thus the books brought by these refugees came as a revelation to Europe, and as we shall see later gave an immense stimulus to the study of science.

CHAPTER VII.

ON SYSTEMS OF NUMERATION*.

I HAVE in many places alluded to the Greek metho expressing numbers in writing, and I have thought it be defer to this chapter the whole of what I wanted to say o various systems of numerical notation which were displ by that of the Arabs.

First as to symbolism and language. The plan of cating numbers by the digits of one or both hands is so na that we find it in universal use among early races, and members of all tribes now extant are able to indicate by numbers at least as high as ten. For larger numbers we however reach a limit beyond which primitive man is un to count, while as far as language goes it is well known many tribes have no word for any number beyond four, that they express all higher numbers by the words plent heap. It is worth remarking that the Egyptians used symbol for the word heap to denote an unknown quanti algebra (see p. 9).

The number five is generally represented by the open h and it is said that in almost all languages the words five hand are derived from the same root. It is possible that in times men did not readily count beyond five, and things if numerous were counted by multiples of it. Thus the Ro

* The subject of this chapter is discussed by Cantor and Hankel; great deal of valuable information is also to be found in the arti *Arithmetic* by Dean Peacock in the *Encyclopædia Metropolitana*, *Sciences*, London, 1845. *Les signes numéraux et l'arithmétique ch peuples de l'antiquité*... by T. H. Martin, Rome, 1864; several ar by A. F. Pott on the subject; and *Die Zahlzeichen*... by G. Frie Erlangen, 1869, should also be consulted.

symbol X for ten probably represents two "V"s, placed apex to apex, and seems to point to a time when things were counted by fives: see also the *Odyssey*, IV. 412—415, which apparently refers to a similar custom. In connection with this it is worth noticing that both in Java and also among the Aztecs a week consisted of five days.

But the members of nearly all races of which we have now any knowledge seem to have used the digits of both hands to represent numbers. They could thus count up to and including ten, and were therefore led to take ten as their radix of notation. In the English language for example all the words for numbers are expressed on the decimal system, except those for 11 and 12: the use of special words for these (instead of one-teen and two-teen, as by analogy they ought to have been), being derived from the common use of a duodecimal subdivision of things among all people of Aryan descent.

Some tribes seem to have gone further and by making use of their toes were accustomed to count by multiples of twenty. The Aztecs, for example, are said to have done so. It may be noticed that we still count some things (e.g. sheep) by scores, the word score signifying a notch or scratch made on the completion of the twenty; while the French also talk of quatre-vingt, as though at one time they counted things by multiples of twenty. I am not, however, sure whether the latter argument is worth anything, for I have an impression that I have seen the word *octante* in old French books; and there is no question that *septante* and *nonante* were at one time the usual words for seventy and ninety (see, for example, de Kempten's Arithmetic, Paris, 1564).

The only tribes of whom I have read who did not count in terms either of five or of some multiple of five are the Bolans of West Africa who are said to have counted by multiples of seven, and the Maories who are said to have counted by multiples of eleven.

Up to ten it is comparatively easy to count; but primitive people found and still find great difficulty in counting higher

numbers; apparently at first this difficulty was only overcome by the method (still in use in South Africa) of getting two men, one to count the units up to ten on his fingers, and the other to count the number of groups of ten so formed. To us it is obvious that it is equally effectual to make a mark of some kind on the completion of each group of ten, but it is said that the members of many tribes never succeeded in counting numbers higher than ten unless by the aid of two men.

Most races who shewed any aptitude for civilization proceeded further and invented a way of representing numbers by means of pebbles or counters arranged in sets of ten: and this in its turn developed into the abacus or swan-pan. This instrument was in use among nations so widely separated as the Etruscans, Greeks, Egyptians, Hindoos, Chinese, and Mexicans; and was it is believed invented independently at several different centres. It is still in common use in Russia, China, and Japan.

In its simplest form (fig. i) the abacus consists of a wooden board with a number of grooves cut in it, or of a table covered with sand in which grooves are made with the fingers. To represent a number, as many counters or pebbles (*calculi*) are put on the first groove as there are units, as many on the second as there are tens, and so on. When by its aid a number of objects are counted, for each object a pebble is put on the first

Fig. 1.

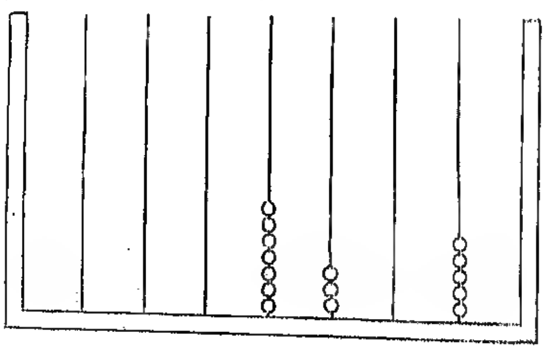


Fig. ii.

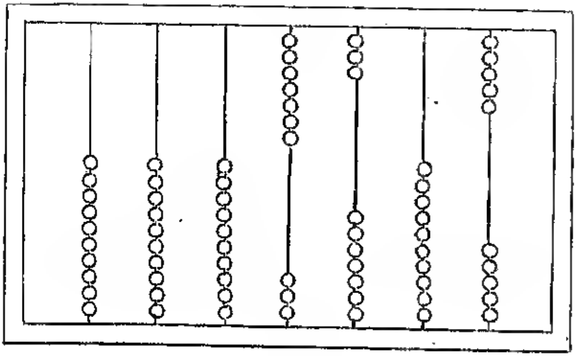
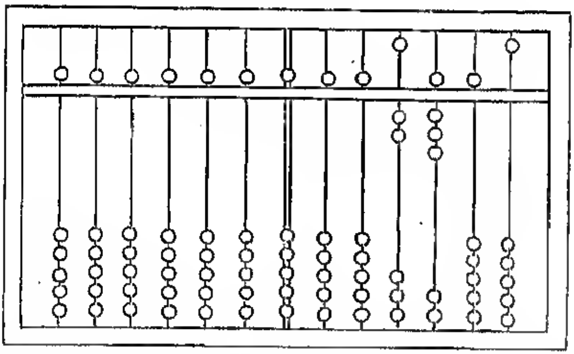


Fig. iii.



groove; and as soon as there are ten pebbles there, they are taken off and one pebble put on the second groove; and so on. It was sometimes, as in the Aztec *quipus*, made with a number of parallel wires or strings stuck in a piece of wood on which beads could be threaded; and in that form is called a swan-pan. In the number, represented in each of the instruments above figured, there are seven thousands, three hundreds, no tens, and five units, i.e. it is 7,305. Some races counted from left

to right, others from right to left, but this is a mere ma of convention.

The Roman abaci seem to have been rather more elabo They contained two marginal grooves or wires, one with beads to facilitate the addition of fractions whose denomina were four, and one with twelve beads for fractions whose nominators were twelve: but otherwise they do not diff principle from those described above. They were gene made to represent numbers up to 100,000,000. There ar Greek abaci now in existence, but there is no doubt they similar to the Roman ones. The Greeks and Romans their abaci as boards on which they played a game somewh like backgammon.

In the Russian *tschotü* (fig. ii) the instrument is impr by having the wires set in a rectangular frame, and ten (or beads are permanently threaded on each of the wires, the being considerably longer than is necessary to hold them the frame be held horizontal, and all the beads be toward side, say the lower side of the frame, it is possible to rep any number by pushing towards the other or upper si many beads on the first wire as there are units in the nu as many beads on the second wire as there are tens i number, and so on. It is easy to see how numbers can be added or subtracted: multiplication and division can be performed by its aid. This form is used in the K garten schools. The calculations are made somewhat rapidly if the five beads on each wire next to the upp are coloured differently to those next to the lower side.

Figure iii represents the form of swan-pan in common China and Japan. There the development is carried on further, and five beads on each wire are replaced by a bead of a different form or on a different division. I a that an expert Japanese can by the aid of it add numb rapidly as they can be read out to him. It will be i that the instrument represented in the figure is made two numbers can be expressed at the same time on it.

The abacus is obviously only a concrete way of representing a number in the decimal system of notation, that is, by means of the local value of the digits. Unfortunately the method of writing numbers developed on different lines, and it was not until about the thirteenth century of our era when a symbol zero used in conjunction with nine other symbols was introduced that a corresponding notation in writing was adopted in Europe.

Next as to the means of representing numbers in writing. In general we may say that from the earliest times a number was (if represented by a sign and not a word) indicated by the requisite number of strokes. These may have been mere marks; or perhaps they represented fingers, since in the Egyptian hieroglyphics the symbols for the numbers 1, 2, 3, are one, two, or three fingers respectively, though in the later hieratic writing they had become reduced to straight lines. Thus in an inscription from Tralles in Caria of the date 398 B.C. the phrase seventh year is represented by ετεος | | | | | | |. Additional symbols for 10 and 100 were soon introduced: and the oldest extant Egyptian and Phœnician writings repeat the symbol for unity as many times (up to 9) as was necessary, and then repeat the symbol for ten as many times (up to 9) as was necessary, and so on. No specimens of Greek numeration of a similar kind are in existence, but there is every reason to believe the testimony of Iamblichus who asserts that this was the method in which the Greeks first expressed numbers in writing.

This way of representing numbers remained in current use throughout Roman history; and for greater brevity they or the Etruscans added separate signs for 5, 50, &c. The Roman symbols are generally merely the initial letters of the names of the number; thus C stood for centum or 100, M for mille or 1000. The symbol V for 5 seems to have originally represented an open palm with the thumb extended; just as our present symbol for zero, 0, is supposed to represent a closed hand. The symbols L for 50 and D for 500 are said to represent the upper halves of the symbols used in early times

for C and M. The subtractive forms like IV for IIII are probably of a later origin.

Similarly in Attica five was denoted by Π or Γ for πέντε, ten by Δ for δέκα, a hundred by H for ἑκατόν, a thousand by X for χίλιοι: while 50 was represented by a Δ written inside a Π, and so on. These Attic symbols continued to be used for inscriptions and formal documents until a late date.

This, if a clumsy, is a perfectly intelligible system; but the Greeks at some time in the third century before Christ abandoned it for one which offers no special advantages in denoting a given number, while it makes all the operations of arithmetic exceedingly difficult. In this, which is known from the place where it was introduced as the Alexandrian system, the numbers from 1 to 9 are represented by the first nine letters of the alphabet; the tens from 10 to 90 by the next nine letters; and the hundreds from 100 to 900 by the next nine letters. To do this the Greeks wanted 27 letters, and as their alphabet only contained 24, they reinserted two letters (the digamma and koppa) which had formerly been in it but had become obsolete, and introduced at the end another symbol taken from the Phœnician alphabet. Thus the ten letters α to ι stood respectively for the numbers from 1 to 10; the next eight letters for the multiples of 10 from 20 to 90; and the last nine letters for 100, 200, &c. up to 900. Intermediate numbers like 11 were represented as the sum of 10 and 1, that is by the symbol ια′. This afforded a notation for all numbers up to 999; and by a system of suffixes and indices it was extended so as to represent numbers up to 100,000,000.

There is no doubt that these signs were at first only used as a way of expressing a result attained by some concrete or experimental method, and the idea of operating with the symbols themselves in order to obtain the results is of a later growth and is one with which the Greeks never became familiar. The non-progressive character of Greek arithmetic may be partly due to their unlucky adoption of the Alexandrian system which caused them for most practical purposes

to rely on the abacus and to supplement it by a table of multiplications which was learnt by heart. The results of the multiplication or division of numbers other than those in the multiplication table might have been obtained by the use of the abacus, but in fact they were generally got by repeated additions and subtractions. Thus as late as 944 a certain mathematician who in the course of his work wants to multiply 400 by 5 finds the result by addition. The same writer when he wants to divide 6152 by 15 tries all the multiples of 15 until he gets to 6000, this gives him 400 and a remainder of 152; he then begins again with all the multiples of 15 until he gets to 150, and this gives him 10 and a remainder of 2. Hence the answer is 410 with a remainder 2.

A few mathematicians however such as Hero of Alexandria, Theon, and Eutocius multiplied and divided in what is essentially the same way as we do. Thus to multiply 18 by 13 they proceeded as follows:

$$
\begin{array}{llll}
\iota\gamma \times \iota\eta = & (\iota+\gamma)(\iota+\eta) & 13 \times 18 = & (10+3)(10+8) \\
 = & \iota(\iota+\eta)+\gamma(\iota+\eta) & = & 10(10+8)+3(10+8) \\
 = & \rho+\pi+\lambda+\kappa\delta & = & 100+80+30+24 \\
 = & \sigma\lambda\delta & = & 234.
\end{array}
$$

I suspect that the last step, in which they had to add four numbers together, was obtained by the aid of the abacus.

These however were men of exceptional genius, and we must recollect that for all ordinary purposes the art of calculation was performed with the aid of the abacus and the multiplication table only, while the term arithmetic was confined to the theories of ratio, proportion, and of numbers (see p. 63).

SECOND PERIOD.

THE MATHEMATICS OF THE MIDDLE AGES AND OF THE RENAISSANCE.

This period begins about the sixth century, and may be said to end with the invention of analytical geometry and of the infinitesimal calculus. The characteristic feature of this period is the creation of modern arithmetic, algebra, and trigonometry.

I commenced this history by dividing it into three periods. I have discussed the history of mathematics under Greek influence, and I now come to that of the mathematics of the middle ages and renaissance. I shall consider first in chapter VIII. the rise of learning in western Europe, and the mathematics of the middle ages. Next in chapter IX. I shall discuss the nature and history of Arabian mathematics, and in chapter X. their introduction into Europe. I shall then in chapter XI. trace the subsequent progress of arithmetic to the year 1637. Next in chapter XII. I shall treat of the general history of mathematics during the renaissance from the invention of printing to the commencement of the seventeenth century, say from 1450 to 1637; which is really an account of the development of arithmetic, algebra, and trigonometry. Lastly in chapter XIII. I shall consider the revival of interest in mechanics, experimental methods, and pure geometry which marks the last few years of this period, and serves as a connecting link between the mathematics of the renaissance and those of modern times.

CHAPTER VIII.

THE RISE OF LEARNING IN WESTERN EUROPE. CIRC. 600—1200.

SECTION 1. *Education in the sixth, seventh, and eighth centuries.*
SECTION 2. *The Cathedral and Conventual schools.*
SECTION 3. *The rise of the early medieval universities.*

Education in the sixth, seventh, and eighth centuries.

THE first few centuries of this second period of our history are singularly barren of any interest; and indeed it would be strange if we found any knowledge of either science or mathematics among those who were just emerging from barbarism and lived in a condition of perpetual war. Broadly speaking we may say that from the sixth to the eighth centuries all that survived of ancient thought in western Europe was preserved in the Benedictine monasteries. We may there find some slight attempts at a study of literature, but mathematics was never read: to learn the use of the abacus, to keep accounts, and to know the rule by which the date of Easter could be determined was all the science that the most studious aimed at. Nor was this unreasonable, for the monk had renounced the world, and there was no reason why he should learn more science than was required for the services of the church or his monastery.

When, in the latter half of the eighth century, Charles the Great had established his empire, he determined to do what he could to remedy the evil; and he began by commanding that schools* should be opened in connection with every cathedral

* See *The schools of Charles the Great and the restoration of education in the ninth century* by J. B. Mullinger, London, 1877.

and monastery in his kingdom; an order which was approved and materially assisted by the popes. It is interesting to us to know that this was done at the instance and under the direction of two Englishmen, Alcuin and Clement, who had attached themselves to his court; a fact which may serve to remind us that during the eighth century England and Ireland were in advance of the rest of Europe as far as learning went.

Of these the more prominent was Alcuin,* who was born in Yorkshire in 735 and died at Tours in 804. He was educated at York under Archbishop Egbert, his "beloved master," whom he succeeded as director of the school there. Subsequently he became abbot of Canterbury, and was sent to Rome by Offa to procure the *pallium*. On his journey back, he met Charles at Parma; the emperor took a great liking to him, and finally induced him to take up his residence at the imperial court, and there teach rhetoric, logic, mathematics, and divinity. Alcuin remained for many years one of the most intimate and influential friends of Charles, who constantly employed him as a confidential ambassador; as such he spent the years 791 and 792 in England, and while there reorganized the studies at his old school at York. In 801 he begged permission to retire from the court so as to be able to spend the last years of his life in quiet: with difficulty he obtained leave, and went to the Abbey of S. Martin at Tours, of which he had been made head in 796. He established a school in connection with the abbey which became very celebrated, and he remained and taught there till his death on May 19, 804.

The correspondence and works of Alcuin were published in 2 vols. by Froben at Ratisbon in 1777. Most of them deal with theology or history, but they include a collection of arithmetical propositions suitable for the instruction of the young. The majority of the propositions are easy problems, either determinate or indeterminate, and are, I presume, founded on

* See the life of Alcuin by F. Lorentz, Halle, 1829, translated by J. M. Slee, London, 1837; and *Alcuin und sein Jahrhundert* by K. Werner, Paderborn, 1876.

works with which he had become acquainted when at Rome. The following is one of the most difficult, and will give an idea of the character of the work. If one hundred bushels of corn are distributed among one hundred people in such a manner that each man shall receive three bushels, each woman two, and each child half a bushel: how many men, women and children were there? The general solution is $(20-3n)$ men, $5n$ women, and $(80-2n)$ children, where n may have any of the values 1, 2, 3, 4, 5, 6. Alcuin only gives the solution for which $n=3$; i.e. he says 11 men, 15 women, and 74 children.

This however was the work of a man of exceptional genius, and we shall probably be correct in saying that mathematics, if taught at all, was generally confined to the geometry of Boethius, the use of the abacus and multiplication table, and possibly the arithmetic of Boethius. It was of course natural that the works used should come from Roman sources, for Britain and all the countries included in the empire of Charles had at one time formed part of the western half of the Roman empire and their inhabitants continued for a long time to regard Rome as the centre of civilization, while the higher clergy kept up a tolerably constant intercourse with Rome.

The Cathedral and Conventual schools.

After the death of Charles, most of the schools confined themselves to teaching Latin and music, that is those subjects some knowledge of which was essential to the worldly success of the higher clergy. This was not much, but the continued existence of the schools gave an opportunity to any teacher whose learning or zeal exceeded those narrow limits; and though there were but few who availed themselves of the opportunity, yet the number of those desiring instruction was so large that it would seem as if any one who could teach was certain to attract a considerable audience. A few schools at which this was the case became large, and acquired a certain degree of permanence. The subjects taught were still confined

within the narrow limits of the preliminary trivium, viz. grammar, logic, and rhetoric, which practically meant the art of reading and writing Latin; and the advanced quadrivium, viz. arithmetic sufficient to enable one to keep accounts, music for the church services, geometry for the purpose of land surveying, and astronomy sufficient to enable one to calculate the feasts and fasts of the church. The seven liberal arts are enumerated in the line

Lingua, tropus, ratio; numerus, tonus, angulus, astra.

Any student who got beyond the trivium was looked on as a man of great erudition, *Qui tria, qui septem, qui totum scibile novit*, as a verse of the eleventh century runs. The special questions that then and long afterwards attracted the best thinkers were logic and certain portions of transcendental theology and philosophy. We may sum the matter up by saying that during these centuries the mathematics usually taught was still confined to that comprised in the two works of Boethius together with the practical use of the abacus and the multiplication table, though during the latter part of the time a wider range of reading was undoubtedly accessible.

In the tenth century a man appeared who would in any age have been remarkable and who gave a great stimulus to learning. This was Gerbert*, an Aquitanian by birth, who died in 1003 at about the age of fifty. His abilities attracted attention to him even when a boy, and procured his removal from the abbey school at Aurillac to the Spanish march where he received a good education. He was in Rome in 971, and his proficiency in music and astronomy excited considerable interest: at that time he was not much more than twenty, but he had already mastered all the branches of the trivium and quadrivium, as then taught, except logic; and to

* See *La vie de Gerbert*, by A. Olleris, Clermont, 1867; *Gerbert von Aurillac*, by K. Werner, 2nd Edition, Vienna, 1881; and *Gerbert, Beiträge zur Kenntniss der Mathematik des Mittelalters*, by H. Weissenborn, Berlin, 1888.

learn this he moved to Rheims which the archbishop Adalbero had made the most famous school in Europe. Here he was at once invited to teach, and so great was his fame that Hugh Capet entrusted to him the education of his son Robert, who was afterwards king of France. Gerbert was especially famous for his construction of abaci, and of terrestrial and celestial globes; he was accustomed to use the latter to illustrate his lectures. These globes excited great admiration which he utilized by offering to exchange them for copies of classical Latin works, which seem to have already become very scarce; and the better to effect this he appointed agents in all the chief towns of Europe. To his efforts it is believed we owe the preservation of several Latin works; but he made a rule to reject the Christian fathers and Greek authors from his library. In 982 he received the abbey of Bobbio, and the rest of his life was taken up with political intrigues. He became archbishop of Rheims in 991, and of Ravenna in 998; he was elected pope in 999, when he took the title of Sylvester II.; he at once commenced an appeal to Christendom to arm and defend the Holy Land, thus forestalling Peter the Hermit by a century, but he died on May 12, 1003 before he had time to elaborate his plans. His library is I believe still in the Vatican.

So remarkable a personality left a deep impress on his generation, and all sorts of fables soon began to collect around his memory. It seems certain that he made a clock which was long preserved at Magdeburg, and an organ worked by steam which was still at Rheims two centuries after his death. All this only tended to confirm the suspicions of his contemporaries that he had sold himself to the devil; and the details of his interviews with that gentleman, the powers he purchased, and his efforts to escape from his bargain when he was dying, may be read in the pages of William of Malmesbury, Orderic Vitalis, and Platina. To these anecdotes the former adds the story of the statue inscribed with the words "strike here," which having amused our ancestors in the *Gesta romanorum* has been recently told again in the *Earthly paradise*.

Extensive though his influence was, it must not be supposed that Gerbert's writings show any great originality. His mathematical works comprise a treatise on *the use of the abacus*, one on arithmetic entitled *De numerorum divisione*, and one on *geometry*. The geometry is of very unequal ability; it includes a few applications to land-surveying and the determination of the heights of inaccessible objects, but much of it seems to be copied from some pythagorean text-book. In the course of it he however solves one problem which was of remarkable difficulty for that time. The question is to find the sides of a right-angled triangle whose hypothenuse and area are given. He says, in effect, that if these latter be respectively denoted by c and h^2, then the lengths of the two sides will be

$$\tfrac{1}{2}\{\sqrt{c^2+4h^2}+\sqrt{c^2-4h^2}\} \text{ and } \tfrac{1}{2}\{\sqrt{c^2+4h^2}-\sqrt{c^2-4h^2}\}.$$

One of his pupils **Bernelinus** published a work on the abacus which is, there is very little doubt, a reproduction of the teaching of Gerbert. It is valuable as indicating that the Arabic system of numeration was still unknown in Europe.

The rise of the early mediaeval universities.*

At the end of the eleventh century or the beginning of the twelfth a great revival of learning took place at several of these cathedral or monastic schools; or perhaps we should rather say that in some cases teachers who were not members of the school settled in its vicinity and with the sanction of the authorities gave lectures, which were in fact always either on theology, logic, or civil law. As the students at these centres grew in numbers, it became possible and desirable to act to-

* Nearly all the known facts on the subject of the mediaeval universities are collected in *Die Universitäten des Mittelalters bis 1400* by P. H. Denifle, Berlin, 1885; a work which both in form and matter is typical of German research. See also vol. i. of *The University of Cambridge* by J. B. Mullinger, Cambridge, 1873; and *État des lettres au xiv^e siècle* by V. Le Clerc, 2 vols., Paris (2nd ed.), 1865.

gether whenever any interest common to all was concerned. The association thus formed was a sort of guild or trades union, or in the language of the time a *universitas scholarium*. This was the first stage in the development of every early mediaeval university. In some cases as at Paris the university was formed by the teachers alone, in others as at Bologna by both teachers and students; but in all cases precise rules for the conduct of business and the regulation of the internal economy of the guild were formulated at an early stage in its history. The municipalities and numerous societies which existed in Italy supplied plenty of models for the construction of such rules. We are, almost inevitably, unable to fix the exact date of the commencement of these voluntary associations, but they existed at Paris, Bologna, Salerno, Oxford, and Cambridge before the end of the twelfth century. Whether a loosely associated and self-constituted guild of students such as these were can be correctly described as a university is a doubtful point. These societies all seem to have arisen in connection with schools established by some church or monastery, and I believe every mediaeval university was built up under the protection of the bishop of some see. The guild was thus at first in some undefined manner subject to the special authority of the bishop or his chancellor, from the latter of whom the head of the university subsequently took his title. The schools from which the universities sprang continued for a long time to exist under the direct control of the cathedral or monastic authorities by the side of the guilds formed by the teachers on the more advanced subjects.

The next stage in the development of the university was its recognition by the sovereign of the kingdom in which it was situated. It was generally thus given exclusive jurisdiction over its own students, and allowed to control all municipal regulations which in any way affected its members, and finally it usually received a definite charter of incorporation. Its degrees were then accepted throughout the kingdom. I believe no university was thus acknowledged before the end of

the twelfth century. Paris received its charter in 1200 and was probably the earliest university in Europe. A medical school existed at Salerno as early as the ninth century, and a legal school at Bologna as early as 1138, but most critics consider that the universities to which these schools respectively gave rise must be referred to a later date.

The last step was the acknowledgment of its corporate existence by the pope (or emperor), and the recognition of its degrees as a title to teach throughout Christendom. Paris was thus recognized in 1283. A mediaeval university therefore passed through three stages: first it was a self-constituted guild of students; second, legal privileges were conferred on it by the state and it was usually incorporated; third, it was recognized by the pope and its degrees declared current throughout the whole of Christendom. Such is a general outline of the history of the bodies under which learning was carried on in the middle ages. I add in a footnote a few additional particulars connected with the early history of Paris, Oxford, and Cambridge*.

* Paris is probably the oldest European university, and as it served as the model on which Oxford and Cambridge were subsequently constituted, its history possesses special interest for English readers. The first of these stages in its history may perhaps be dated as far back as 1109 when William of Champeaux began to teach logic, and may certainly be said to have commenced when his pupil Abelard was lecturing on logic and divinity. The faculty of arts and (probably) its form of self-government existed in 1169, for Henry II. proposed to refer his quarrel with Becket to it and two other bodies; it is also alluded to in two decretals of the pope in 1180. By an ordinance of the king of France in 1200 the university entered on the second of these stages and its members were granted exemption from all ordinary tribunals; in 1208 it was incorporated, and thus put on a permanent basis, which its mere recognition by the state did not effect. In 1257 theology, and in 1281 law and medicine, were created separate faculties. About the same time the pope Nicholas IV. decreed that its doctors should enjoy the privileges and rank of doctors throughout Christendom.

The collegiate system also originated in Paris. The religious orders established hostels for their own students about the middle of the twelfth century, and it is possible that S. Thomas' College and the Danish College

The majority of the great mathematicians of subsequent times have been closely connected with one or more of the

in the Rue de la Montagne were founded about 1200; but if we reject these, the dates of their foundation being uncertain, the first regular college was that founded by Robert de Sorbonne in 1250. The college of Navarre which far surpassed all others in wealth and numbers was founded in 1305. Two hundred years later there were 18 large colleges, and 80 hostels or small and generally unendowed colleges; by that time all the large colleges had specialized their higher teaching on some one subject, and all but one had thrown their lectures open to the university, while the smaller colleges had abandoned teaching altogether except in the case of Latin grammar. The want of discipline among the non-collegiate students led to their suppression at an early date.

The first definite body of statutes seems to have been formed in 1208, about the same time as the university was recognized by the crown and incorporated. In 1215 the cardinal legate Robert de Courçon laid down a curriculum, and thenceforward all European universities have imposed a definite course of study combined with certain periodical tests of proficiency, on their junior members. The whole modern system of university education dates from this order.

The internal organization with its different nations, proctors, separate faculties, and various governing bodies which acted as a check one on the other was very complex. But leaving these on one side we may say that in general (and the remark applies equally to Oxford and Cambridge) the final degree of master or doctor was at first merely a license to teach. It was given to any student who had gone through the recognized course of study and shewn he was of good moral character. Outsiders were also admitted, but not as a matter of course. To obtain the degree it was necessary first to get a *licentia docendi*, and secondly to be "incepted," that is admitted by the whole body of teachers or regents as one of themselves. The *licentia docendi* was originally granted on proof of good moral character by the chancellor of the chapter of the church with which the university was in close connection. For inception the student had to keep an act or give a lecture before the whole university, and to be recommended by a master of arts under whom he had studied. On admission to the degree he gave a dinner or presents to his new colleagues: this tax on his resources was subsequently changed into a fee to the university chest. The teaching and government were (except at Bologna) entirely in the hands of the incepted members, who were at first called indifferently masters, regents, doctors, or professors; and no one took a degree who did not intend to reside and teach. A survival of this idea exists in the technical description of a doctor of divinity

universities and the general standard of mathematics has been largely fixed by them. They did not however for a long time

at Oxford and Cambridge as *sacræ theologiæ professor*. The new master was not permitted to exercise his functions until the term after that in which he incepted—a survival of which custom still exists in the Cambridge commencement. By the beginning of the fourteenth century students began to seek for degrees without any intention of teaching, and in 1426 the university of Paris took on itself to refuse a degree to a student—a Slavonian, one Paul Nicolas—who had done badly (see Bulæus vol. v. p. 377). He was I believe the first student who was ever "plucked."

The range of studies varied at different times but usually included the very elements of a liberal education. This seems to have been due to the early age at which some, and perhaps the majority, of the students then entered. The records of Paris show that in the thirteenth century it was not unusual to come into residence at as early an age as that of 11 or 12; while the highest degrees such as doctor of divinity could not be taken before the age of 25, and were in fact rarely taken before 30 or 35. The students were of every age between these limits.

Lastly I would remark that though all the mediæval universities grew up under the protection of some bishop or abbot, and though the bulk of their members were ordained, they were not ecclesiastical organizations; and their connection with the church arose chiefly from the fact that clerks were then the only class of the community who were left free by the state to pursue their studies. This is the explanation of the struggles so successfully waged by every early university against the authority of the bishop and his chancellor on the one hand, and against the proselytizing carried on amongst the students by the monks on the other. The numerous hostels established by the great monastic orders in Paris, Oxford, and Cambridge, are not now considered as having formed any part of those universities.

It would take me beyond my limits if I were to trace the history of the university of Paris farther. Its decay is generally dated from the year 1719. Up to that time a teacher or regent received from his college board, lodging, and sufficient money to enable him to live, but he depended for his luxuries on the fees of those who attended his lectures: hence there was every encouragement to make the lectures efficient. The stipends of the professors also depended to a large extent on their efficiency. This was altered in 1719, and professors whose lectures were gratuitous were subsequently appointed for life at a fixed stipend. Perhaps the 18th century was an unfavourable time for the experiment, but the result was disastrous: those graduates of the colleges, who con-

extend the subjects of instruction, but they carried them to a somewhat higher standard. We may regard the trivium, the

tinued to charge fees, soon found their lecture-rooms deserted: within forty years the number of hostels was reduced to less than 40, and that of the large colleges to 10, most of which were heavily in debt; in 1764 the hostels were shut up; finally on Sept. 15, 1793, the Convention suppressed the university and colleges, and appropriated their revenues. The present university of France with faculties at Paris, Nancy, and other places is a creation of Napoleon I.

The other universities which rival Paris in antiquity are those of Bologna and Salerno which were respectively the great schools of civil law and of medicine in mediaeval Europe, but I dwelt on the history of the university of Paris partly because it is generally taken as the typical mediaeval university, partly because both Oxford and Cambridge were founded by those who were familiar with its constitution.

The towns of Oxford and Cambridge are both minutely described in Domesday book, and as there is not the slightest reference to any school or university we may safely assert that neither existed in the year 1086. It is on the whole probable that schools of some kind had existed there during the Saxon period. But at the latest those at Oxford were destroyed by the Danes in 1009; and those at Cambridge in 1010, when the Danes having sailed up to Ely, on what was then a broad river, pushed on to Cambridge, and burnt it to the ground. All the data we possess would seem to point to the fact that shortly after the Conquest schools at these towns were established in connection respectively with the priory of S. Frideswyde and the conventual church at Ely; but why these particular schools developed into universities we are unable to say.

The first reliable mention of Oxford as a place of education is in 1133 when Robert Pullen came from Paris and lectured on theology. A little later, in 1149, Vacarius came from Bologna and taught civil law. It is not unlikely that the Benedictine monastery of S. Frideswyde was ruled by French monks, and that the lectures were given under their influence and in their monastery: but the references seem to imply that there was then no university there. In 1180 there is an allusion to a scholar in the *Acta Sanctorum* p. 579; and in 1184 Giraldus Cambrensis lectured to the masters and scholars. (*Gir. Camb.* vol. I. p. 23.) Hence it is almost certain that the university had its origin between 1150 and 1180. Mr Rashdall believes that it developed out of a migration from Paris in 1167, but the available data do not seem to me to justify any such definite statement. In 1214 the university was given legal jurisdiction whenever one party was a scholar or the servant of a scholar. In 1244 it was incorporated by Henry III. The collegiate system commenced with the

course for which took four years, as forming the ordinary course of study for all students, and we may represent those who were

foundation of Merton College in 1264; though the money for building University College was left in 1249, and for Balliol College in 1263. The university was recognized by Innocent IV. in 1254, but it was not till 1296 that the masters received from Boniface VIII. permission to teach anywhere in Christendom.

I wish I could be equally explicit about Cambridge, but unfortunately its early records and charters were burnt. All the mediaeval universities were divided into "nations" according to the place of birth of their students. There was a constant feud at Cambridge between those born north of the Trent and those born to the south of it. In 1261 a desperate fight lasting some days took place between the two factions, in the course of which the university records were burnt. A similar disturbance took place in 1322. Again in 1381 in the course of the popular disturbances then prevalent throughout the kingdom a mob of townsmen broke into S. Mary's Church, seized the university chest, and burnt all the charters and documents therein contained. I remark, in passing, that the university church was then and long afterwards the ordinary place of business and meeting of the university, and it would seem that the university has still the right to use it whenever they want, and (apparently) to have the use of the bells of either it or of S. Benet's Church. Nor was there anything unusual in this, for the idea that the use of a church should be confined to religious functions is of modern growth: plays were formerly acted in the chapels of King's and Trinity, and at the latter college elections are still held and leases sealed in the chapel, where also the various declamations are recited. When, after the outbreak of 1381, order was restored the liberties of the town were confiscated and bestowed on the vice-chancellor in whom they remained vested till the reign of Henry VIII. To this stringent measure the subsequent prosperity of the university (and so indirectly of the town) was largely due.

The original charters having been destroyed, we are compelled in their absence to rely on allusions to them in trustworthy authorities. Now it was the custom at both universities to solicit a renewal of their privileges at the beginning of each reign (an opportunity of which they often took advantage to get them extended) and it is possible that the dates here given may be those of the renewals of original charters which are now lost. At any rate it would seem certain that the university existed in its first stage, i.e. as a self-constituted and self-governing community, before 1209, as several students from Oxford migrated in that year to the university of Cambridge; and it is clear it did not exist

studying it as school-boys. The degree of bachelor of arts was given at the end of that course, and the title only meant that

in 1112 when the canons of S. Giles opened schools at their new priory at Barnwell. It was at some time then between those two dates that the university entered on its first stage of existence. In 1225 there is an allusion to the chancellor of the university in some legal proceedings (Record office, *Coram Rege Rolls* Hen. III. Nos. 20 and 21). In 1229 after some disturbances in Paris, Henry III. invited French students to come and settle at Oxford or Cambridge, and some hundreds came to Cambridge. In 1231 Henry III. gave the university jurisdiction over certain classes of townsmen, in 1251 he extended it so as to give exclusive legal jurisdiction in all matters concerning scholars, and finally confirmed all its rights in 1260. [These privileges were given by letters and enactments, and the first charter of which we now know anything was that given by Edward I. in 1291.] The collegiate system commenced with the foundation of what was afterwards known as Peterhouse in or before 1280. The university was recognized by letters from the pope in 1233, but in 1318 John XXII. gave it all the rights which were or could be enjoyed by any university in Christendom. Under these sweeping terms it obtained exemption from the jurisdiction both of the bishop of Ely and the archbishop of Canterbury (as settled in the Barnwell process, 1430).

Just as the old monastic schools continued to exist by the side of the university of Paris, so the grammar schools, which had originally attracted students to Cambridge and from which the university may therefore be said to have sprung, continued to exist up to the sixteenth century. There would seem to have been nearly a dozen such schools in the thirteenth century, each under one master, and all under the supervision of the *magister glomeriae*. This master of glomery was appointed by the archdeacon of Ely, which strengthens the view that the glomery schools were originally founded by the conventual church at Ely. He was a member of the university and had a bedell to attend him, the "glomerels" were of course subject to his authority, but they were not admitted as members of the university. Disputes between the glomerels and the students would seem to have been decided by the regents (who were the governing body of the university) subject to an appeal to the chancellor. These rules were laid down by Hugh Balsham the bishop of Ely in 1275, who decided that the students of the university were in no way subject to the master of glomery.

To these glomerels the university gave the degree of "master in grammar," which served as a licence to teach Latin, and gave the coveted prefix of *dominus* or *magister* (which in common language was generally rendered *dan*, *dom*, or *sir*) and distinguished the clerk from a mere "hedge-

the student was no longer a schoolboy, and therefore in pupilage. A bachelor could not take pupils, could only teach under special restrictions, and probably occupied a position closely

priest." To get this degree the glomerel had not only to shew that he had studied Priscian in the original, but to give a practical demonstration of his proficiency in the mechanical part of his art. Stokys, who was a fellow of King's and registrary of the University, and had for many years been esquire bedell, has left a complete account of the ceremonies of the university about the year 1500, and from this we learn that on the glomerel proceeding to his degree—"then shall the Bedell purvay for every master in Gramer a shrewde Boy, whom the master in Gramer shall bete openlye in the Scolys, and the master in Gramer shall give the Boye a Grote for hys labour, and another Grote to hym that provydeth the Rode and the Palmer, etcetera, *de singulis*. And thus endythe the Acte in that Facultye." The university presented the new master in grammar with a palmer, that is a ferule, and he was then free of the exercise of his profession. The last degree in grammar was given in 1542. This degree is not spoken of as anything exceptional, and I suppose Paris and Oxford conferred similar degrees, but I do not know of any record of them.

Two attempts were made to establish other universities in England. After the riots in 1261 many students from Cambridge went to Northampton, where they were joined by others from Oxford. They even obtained a licence from Henry III. constituting them a university, but having in 1264 taken the side of the barons in the civil war, they were immediately ordered to return to their respective universities.

A more serious attempt was made in 1301 (?) when numerous students from Oxford established themselves by the side of the great Carmelite monastery at Stamford; there some students from Cambridge joined them, and several colleges and halls were founded. The university of Oxford in 1335 induced Edward III. to suppress this attempt; and in 1334 Oxford and Cambridge bound their regents "never to teach anywhere, as in a university, except at Oxford or Cambridge, nor to acknowledge as legitimate regents those who had 'commenced' in any other town in England." An oath to this effect was exacted till about 1855, when it was abolished to meet the wishes of some who desired to assist the university of Durham which had been founded in 1832.

We can express these results in a tabular form thus:

	Paris	*Oxford*	*Cambridge*
In existence before the year	1100	1161	1209
Legal privileges conferred by the state	1200	1214	1231
Foundation of first college	1250	1264	1284
Degrees current throughout Christendom	1283	1296	1318

analogous to that of an undergraduate now-a-days. Some few bachelors then proceeded to the study of civil law, but it was assumed in theory that all went on first through

It will be noticed that the development of these three universities was very similar. This was partly due to the fact that the earliest constitution and regulations of Oxford and Cambridge were copied with almost slavish fidelity from those of Paris. The English formed one of the four nations of the faculty of arts in Paris, and until the hundred years war students of Oxford and Cambridge frequently migrated for a few years to Paris. Migration between Oxford and Cambridge was common in the thirteenth century and was recognized by both universities.

Up to the Reformation Oxford was the wealthier and more influential, but recent critics seem to think that there was little or no difference in the numbers or organization of the two. The mediaeval reputation of Oxford rested chiefly on the schoolmen that it produced in the fourteenth century, and during that century it was the foremost university in Europe. Of its preeminence then its great wealth (its revenues are even now more than a third greater than those of Cambridge) is an interesting and not unpleasant souvenir. In the fifteenth century, with the close of the English wars in France, Paris regained its supremacy while Oxford was almost deserted. The numbers at Cambridge did not fluctuate so violently.

The Reformation was almost wholly the work of Cambridge divines. Since then Cambridge has been rather the larger of the two. The following table, which gives the number of students who took the bachelor degree in the years specified, will exemplify this.

	1501	1559	1601	1650	1700	1750	1800	1850	1880
Oxford		26	103	93				805	612
Cambridge	30	37	162	221	131	106	102	860	780

The numbers for Oxford to 1650 are taken from Wood's MSS. in the Ashmolean Museum. Those for Cambridge to 1650 from the Sloane MSS. in the British Museum. One would have supposed that the numbers for Oxford in the years 1700, 1750 and 1800 could have been easily ascertained but I have been unable to get them. The number of undergraduates in any year after 1600 may be taken roughly as being from four to five times the number of those who took the B.A. I should like to have added the corresponding numbers for the university of Paris, but I do not know where they are to be found.

the quadrivium, the course for which took three years and which included about as much science as was to be found in the pages of Boethius and Isidorus, and then to theology. The subtleties of the scholastic theology and logic, which were the favorite intellectual pursuit of these centuries, may seem to us dreary and barren, but it is only just to say that they afforded an intellectual exercise which fitted men at a later time to develope science, and were certainly a marked advance on what had been previously taught. We must also give the schoolmen the credit of making the Romance tongues both flexible and precise.

We have now arrived at a time when the results of Arab and Greek science became known in Europe. The history of Greek mathematics has been already discussed; I must now temporarily leave the subject of mediæval mathematics, and trace the development of the Arabian schools to the same date; and I must then explain how the schoolmen became acquainted with the Arab and Greek text-books, and how their introduction affected the progress of European mathematics.

CHAPTER IX.

THE MATHEMATICS OF THE ARABS.

SECTION 1. *Extent of mathematics obtained from Greek sources.*
SECTION 2. *Extent of mathematics obtained from the (Aryan) Hindoos.*
SECTION 3. *The development of mathematics in Arabia.*

THE story of Arabian mathematics* is known to us in its general outlines, but we are as yet unable to speak with certainty on many of its details. It is however quite clear that while part of the early knowledge of the Arabs was derived from Greek sources, part was obtained from Hindoo works; and that it was on those foundations that Arab science was built. I will begin by considering in turn the extent of mathematical knowledge derived from these sources.

* The subject is discussed at length by Cantor, by Hankel, and by Kremer in *Kulturgeschichte des Orients unter den Chalifen*, Vienna, 1877. The materials for this chapter are largely derived from several articles by L. A. Sedillot and Fr. Woepcke, of which the most important are *Matériaux pour servir à l'histoire comparée des sciences mathématiques chez les Grecs et les Orientaux*, by L. A. Sedillot, Paris, 1845—9; and the following five articles by Fr. Woepcke, *Sur l'emploi des chiffres Indiens par les Arabes*; *Sur l'histoire des sciences mathématiques chez les Orientaux* (2 articles), Paris, 1855; *Sur l'introduction de l'arithmétique Indien en occident*, Rome, 1859; and *Mémoire sur la propagation des chiffres Indiens*, Paris, 1863.

Extent of mathematics obtained from Greek sources.

According to their traditions, in themselves very probable, the scientific knowledge of the Arabs was at first derived from the Greek doctors who attended the caliphs at Bagdad. It is said that when the Arabian conquerors settled in towns they became subject to diseases which had been unknown to them in their life in the desert; the study of medicine was then almost confined to those Greeks who read natural philosophy, and many of the latter, encouraged by the caliphs, settled at Bagdad, Damascus, and other cities. Their knowledge of all branches of learning was far more extensive and accurate than that of the Arabs, and the teaching of the young, as has often happened in similar cases, soon fell into their hands. The introduction of European science was rendered the more easy as various small Greek schools existed in the countries subject to the Arabs: there had for many years been one at Edessa among the Nestorian Christians, and there were others at Antioch, Emesa, and even at Damascus which had always preserved the traditions and some of the results of Greek learning.

The Arabs soon remarked that the Greeks rested their medical science on the works of Hippocrates, Aristotle, and Galen; and these books were translated into Arabic by order of the caliph Haroun Al Raschid about the year 800. The translations excited so much interest that his successor Al Mamun (813—833) sent a commission to Constantinople to obtain copies of as many scientific works as was possible, while an embassy for a similar purpose was also sent to India. At the same time a large staff of Syrian clerks was engaged whose duty it was to translate the works so obtained into Arabic and Syriac. To disarm fanaticism these clerks were at first termed the caliph's doctors, but in 851 they were formed into a college, and their most celebrated member Honein ibn Ishak was made its first president by the caliph Mutawakkil (847—861). Honein and his son Ishak ibn Honein revised all the translations before they were finally issued. Neither of them knew

much mathematics, and several blunders were made in the works issued on that subject, but another member of the college Tabit ibn Korra shortly published fresh editions which thereafter became the standard texts.

In this way before the end of the ninth century the Arabs obtained translations of the works of Euclid, Archimedes, Apollonius, Ptolemy, and others; and in some cases these editions are the only copies of the books now extant. It is curious as indicating how completely Diophantus had dropped out of notice that as far as we know the Arabs got no manuscript of his great work till 150 years later, and after they had independently established the foundations of algebra.

Extent of mathematics obtained from Hindoo sources.

The Arabs had considerable commerce with India, and a knowledge of one or both of the two great original Hindoo works on algebra had been thus obtained in the caliphate of Al Mansur (754–775), though it was not until fifty or sixty years later that they attracted much attention. The algebra and arithmetic of the Arabs were largely founded on these treatises, and I therefore devote this section to the consideration of Hindoo mathematics.

The Hindoos, like the Chinese, have pretended that they are the most ancient people on the face of the earth, and that to them all sciences owe their creation. But it would appear from all recent investigations that these pretensions have no foundation; and in fact no science or useful art (except a rather fantastic architecture and sculpture) can be traced back to the inhabitants of the Indian peninsula prior to the Aryan invasion. This invasion seems to have taken place at some time in the latter half of the fifth century, or in the sixth century after Christ, when a tribe of the Aryans entered India by the north-west frontier and established themselves as rulers over a large part of the country. Their descendants, wherever they have kept their blood pure, may still be recog-

nized by their superiority over the races they originally conquered; but like the modern Europeans they found the climate very trying, and gradually degenerated. For the first two or three centuries they however retained their intellectual vigour and produced one or two writers of great ability.

The first of these is **Arya-Bhatta**, who lived at Patna, somewhere about 530. He is frequently quoted by Brahmagupta, and in the opinion of some commentators he created algebraic analysis. The only work of his with which we are acquainted is his *Aryabhathiyam* which consists of the enunciations of various rules and propositions written in verse. There are no proofs, and the language is so obscure and concise that it long defied all efforts to translate it. The book is divided into four parts: of these three are devoted to astronomy and the elements of spherical trigonometry; the remaining part contains the enunciations of 33 rules in arithmetic, algebra, and plane trigonometry. In algebra Arya-Bhatta gives the sum of the first, second, and third powers of the first n natural numbers; the general solution of a quadratic equation; and the solution in integers of certain indeterminate equations of the first degree. In trigonometry he gives a table of natural sines of the angles in the first quadrant, proceeding by multiples of $3\frac{3}{4}^\circ$, defining a sine as the semichord of double the angle. Assuming that for the angle $3\frac{3}{4}^\circ$ the sine is equal to the circular measure, he takes for its value 225, i.e. the number of minutes in the angle. He then enunciates a rule which is nearly unintelligible but is probably the equivalent of the statement

$$\sin (n+1)\alpha - \sin n\alpha = \sin n\alpha - \sin (n-1)\alpha - \sin n\alpha \operatorname{cosec} \alpha,$$

where α stands for $3\frac{3}{4}^\circ$, and working with this formula* he constructs a table of sines, and finally finds the value of sin 90°

* The correct formula is

$$\sin (n+1)\alpha - \sin n\alpha = \sin n\alpha - \sin (n-1)\alpha - 4 \sin n\alpha \sin^2 \tfrac{1}{2}\alpha.$$

Arya-Bhatta therefore considered $4 \sin^2 \frac{1}{2}\alpha$ as equal to cosec α, i.e.

$$2 \sin \alpha = 1 + \sin 2\alpha.$$

Using his values of $\sin \alpha$ and $\sin 2\alpha$ this reduces to $2\,(225) = 1 + 449$.

to be 3438. This result is correct if we take 3·1416 as the value of π, and it is interesting to note that this is the number he does in another place assign for it. There is no direct evidence that Arya-Bhatta was acquainted with the decimal system of numeration, and there is no trace in his work of any knowledge of it. Such geometrical propositions as he gives are wrong.

The *Aryabhathiyam* was published in Sanscrit by Kern at Leyden in 1874; but a French translation by Rodet of that part which deals with algebra and trigonometry was issued at Paris in 1879.

The next Hindoo writer of any note is **Brahmagupta**, who is said to have been born in 598 and was probably alive about 660. He wrote a work in verse entitled *Brahma-Sphuta-Siddhanta*, that is the *Siddhanta* or system of Brahma in astronomy. In this two chapters (chaps. XII. and XVII.) are devoted to arithmetic, algebra and geometry; these were translated by H. Colebrooke and published at London in 1817.

The arithmetic is entirely rhetorical. Most of the problems are worked out by the rule of three, and a large proportion of them are on the subject of interest.

In his algebra, which is also rhetorical, he works out the fundamental propositions connected with an arithmetical progression; solves a quadratic equation (but only gives the positive value to the radical); and finds a solution in integers of several indeterminate equations of the first degree, using the same method as that now practised. He gives one indeterminate equation of the second degree, viz. $nx^2+1=y^2$, and gives as its solution $x=2t/(t^2-n)$ and $y=(t^2+n)/(t^2-n)$, but does not explain the process by which he arrives at it. Curiously enough this equation was sent by Fermat as a challenge to Wallis and Lord Brouncker in the seventeenth century, and the latter found the same solutions as Brahmagupta had previously done. It is perhaps worth noticing that the early algebraists, whether Greeks, Hindoos, Arabs, or Italians, drew no distinction between the problems which led to determinate

and those which led to indeterminate equations. It was only after the introduction of syncopated algebra that attempts were made to give general solutions of equations, and the difficulty of giving such solutions of indeterminate equations other than those of the first degree has led to their practical exclusion from elementary algebra.

In geometry Brahmagupta proved the pythagorean property of a right-angled triangle (Euc. I. 47). He gave expressions for the area of a triangle and of a quadrilateral inscribable in a circle in terms of their sides: and shewed that the area of a circle was equal to that of a rectangle whose sides were the radius and semiperimeter. He was less successful in his attempt to rectify a circle, and his result is equivalent to taking $\sqrt{10}$ for the value of π. He also determined the surface and volume of a pyramid and cone: problems over which Arya-Bhatta had blundered badly. The next part of his geometry is almost unintelligible, but it seems to be an attempt to find expressions for several magnitudes connected with a quadrilateral inscribed in a circle in terms of its sides. Most of this is wrong.

He concluded the chapter with the following celebrated problem. "Two anchorites lived at the top of a cliff of height h, whose base was distant mh from a neighbouring village. One descended the cliff and walked to the village, the other flew up a height x and then flew in a straight line to the village. The distance traversed by each was the same. Find x." Brahmagupta gave the correct answer, namely

$$x = mh/(m+2).$$

It must not be supposed that in the original work all the propositions which deal with any one subject are collected together, and it is only for convenience that I have tried to arrange them in that way. It is impossible to say whether the whole of Brahmagupta's results given above are original. He was certainly acquainted with Arya-Bhatta's work, for he reproduces the table of sines there given, but there seems no reason to doubt that the bulk of the algebra and arith-

metic is original: the origin of the geometry is more doubtful.

To make this account of Hindoo mathematics complete I may depart from the chronological arrangement, and say that the remaining great Indian mathematician was Bhaskara who was born in 1114. He would seem to have been the sixth in succession from Brahmagupta as head of an astronomical observatory at Oudjein. He wrote an astronomy of which only four chapters have been translated. Of these one termed *Lilavati* is on arithmetic: this was translated by J. Taylor, Bombay, 1816; and also by H. Colebrooke, London, 1817. A second termed *Bija ganita* is on algebra; this was translated by E. Strachey, London, 1813; and also by H. Colebrooke, London, 1817. The third and fourth are on astronomy and the sphere; these were edited by L. Wilkinson, Calcutta, 1842. This work was I believe known to the Arabs almost as soon as it was written and influenced their subsequent writings, though they failed to utilize or extend most of the discoveries contained in it. The results thus became indirectly known in the West before the end of the twelfth century, but the text itself was not introduced into Europe till within recent times.

The treatise is in verse, but there are explanatory notes in prose. It is not clear whether it is original or whether it is merely an exposition of the results then known in India; but in any case it is most probable that Bhaskara was acquainted with the Arab works which had been written in the tenth and eleventh centuries, and with the results of Greek mathematics as transmitted through Arabian sources. The algebra is syncopated and almost symbolic, which marks a great advance over that of Brahmagupta and of the Arabs. The geometry is also superior to that of Brahmagupta, but this is apparently due to the knowledge of various Greek works obtained through the Arabs.

The first book or *Lilavati* commences with a salutation to the god of wisdom. The general arrangement of the work may be gathered from the following table of contents. Systems

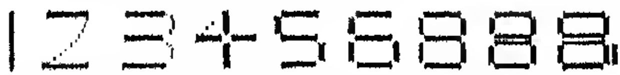
of weights and measures. Next decimal numeration, briefly described. Then the eight operations of arithmetic, viz., addition, subtraction, multiplication, division, square, cube, square-root, and cube-root. Reduction of fractions to a common denominator, fractions of fractions, mixed numbers, and the eight rules applied to fractions. The "rules of cipher"; viz., $a \pm 0 = a$, $0^2 = 0$, $\sqrt{0} = 0$, $a \div 0 = \infty$. The solution of some simple equations which are here treated as questions of arithmetic. The rule of false position. Simultaneous equations of the first degree with applications. Solution of a few quadratic equations. Rule of three, and compound rule of three with various cases. Interest, discount, and partnership. Time of filling a cistern by several fountains (a practical matter to those who used the clepsydra). Barter. Arithmetical progressions, and sums of squares and cubes. Geometrical progressions. Problems on triangles and quadrilaterals. Approximate value of π. Some trigonometrical formulae. Contents of solids. Indeterminate equations of the first degree. Lastly the book ends with a few questions on combinations.

To sum the matter up briefly it may be said that the *Lilavati* gives the rules now current for addition, subtraction, multiplication, and division, as well as the more common processes in arithmetic: while the greater part of the work is taken up with the discussion of the rule of three, which is divided into direct and inverse, simple and compound, and is used to solve numerous questions chiefly on interest and exchange—the numerical questions being expressed in the decimal system of notation.

Bhaskara was celebrated as an astrologer no less than as a mathematician. He learnt by this art that the event of his daughter Lilavati marrying would be fatal to himself. He therefore declined to allow her to leave his presence; but by way of consolation, he not only called the first book of his work by her name, but propounded most of his problems in the form of questions addressed to her. For example: "Lovely and dear Lilavati, whose eyes are like a fawn's, tell me what

are the numbers resulting from 135 multiplied by 12? If thou be skilled in multiplication, whether by whole or by parts, whether by division or by separation of digits, tell me auspicious damsel what is the quotient of the product when divided by the same multiplier?"

This is the earliest known work which contains a systematic exposition of the decimal system of numeration. It is indeed almost certain that Brahmagupta was acquainted with it, and we know that it was in use in India in the year 669, that is in his life-time; but in Bhaskara's arithmetic we meet with the arabic or indian numerals and a sign for zero as part of a well-recognized notation. It is impossible at present to trace these numerals further back than the seventh century, but it is probable that they are at the outside not older than the second or third century after Christ, and there is some evidence to shew that the Aryans obtained them in the first instance from Thibet. The subject is however one of great difficulty, and I only give the above as what seems to me most probable. It has been suggested that the symbols for the first nine numbers were originally formed by drawing as many vertical or horizontal strokes as there are in the number represented in the manner shown below. I conjecturally add dotted lines to make the writing cursive. It will be noticed that the



symbols for seven, eight, and nine are, if written rapidly, almost indistinguishable; and this may account for the introduction of other symbols for seven and nine. The one actually adopted for seven occurs in the Nana Ghat inscriptions in India, circ. 300 B.C., and in most of the Hindoo systems of a later date; the symbol for nine may be derived from that for seven by the addition of two strokes. This conjecture is ingenious, but I am not aware of any historical basis for it.

I may add here that the problems in the Indian works give a great deal of interesting information about the social and economic condition of the country in which they were written. Thus Bhaskara discusses some questions on the price of slaves, and incidentally remarks that a female slave was generally supposed to be most valuable when 16 years old, and subsequently to decrease in value in inverse proportion to the age; thus if when 16 years old she was worth 32 nishkas, her value when 20, would be represented by $(16 \times 32) \div 20$ nishkas. It would appear that, as a rough average, a female slave of 16 was worth about 8 oxen which had worked for two years. The interest charged for money in India varied from $3\frac{1}{2}$ to 5 per cent. per month. Amongst other data thus given will be found the price of provisions and labour.

The chapter termed *Bija ganita* commences with a sentence so ingeniously framed that it can be read as the enunciation either of a religious or a philosophical or a mathematical truth. Bhaskara after alluding to his *Lilavati* or arithmetic states that he intends in this book to proceed to the general operations of analysis. The idea of the notation is as follows. Abbreviations and initials are used for symbols; subtraction is indicated by a dot; addition by juxtaposition merely; but no symbols are used for multiplication, equality, or inequality, these being written at length. A product is denoted by the first syllable of the word subjoined to the factors, between which a dot is sometimes placed. In a quotient or fraction the divisor is written under the dividend without a line of separation. The two sides of an equation are written one under the other, confusion being prevented by the recital in words of all the steps which accompany the operation. Various symbols for the unknown quantity are used, but most of them are the initials of names of colours, and the word colour is often used as synonymous with unknown quantity; its Sanscrit equivalent also signifies a letter, and letters are sometimes used, either from the alphabet, or from the initial syllables of subjects of the problem. In one or two cases symbols are used for the

given as well as for the unknown quantities. The initials of the words square and solid denote the second and third powers, and the initial syllable of square root marks a surd. Polynomials are arranged in powers, the absolute quantity being always placed last and distinguished by an initial syllable denoting known quantity. Most of the equations have numerical coefficients and the coefficient is always written after the unknown quantity. Positive or negative terms are indiscriminately allowed to come first; and every power is repeated on both sides of an equation, with a zero for the coefficient, when wanted. After explaining his notation Bhaskara goes on to give the rules for addition, subtraction, multiplication, division, squaring, and extracting the square root of algebraical expressions; he then gives the rules for the cipher as in the *Lilavati*; solves a few equations; and lastly concludes with some operations on surds.

Other chapters on algebra, trigonometry, and geometrical applications exist, and fragments of them have been translated by Colebrooke. Amongst the trigonometrical formulae is one which is equivalent to the equation $d(\sin\theta) = \cos\theta\, d\theta$ (Delambre, I. 456).

I have departed from the chronological order in treating here of Bhaskara, but as he was the only remaining Hindoo writer of any eminence, I thought it better to mention him at the same time as I was discussing his compatriots. It must however be remembered that he flourished subsequently to all the Arab mathematicians considered in the next section. The works with which the Arabs first became acquainted were those of Arya Bhatta and Brahmagupta, and it is doubtful if they ever made much use of the great treatise of Bhaskara.

It is probable that attention was called to the works of the first two of these writers by the fact that the Arabs adopted the Indian system of arithmetic, and were thus led to look at the mathematical text-books of the Hindoos. The Arabs had always had considerable commerce with India, and with the establishment of their empire the amount of trade naturally

increased. They then, circ. 700, found the Hindoo merchants beginning to use the system of numeration with which we are familiar and adopted it at once. This immediate acceptance of it was made the easier as they had no collection of science or literature written in another system, and it is doubtful whether at that time they possessed any but the most primitive system of notation for expressing numbers. The earliest definite date which I can assign for the use in Arabia of the decimal system of numeration is 773. In that year some Indian astronomical tables were brought to Bagdad, and it is almost certain that in these Indian numerals (including a zero) were used.

The development of mathematics in Arabia.

In the preceding sections of this chapter I have indicated the two sources from which the Arabs derived their knowledge of mathematics, and sketched out roughly the amount of knowledge obtained from each. We may sum the matter up by saying that before the end of the eighth century the Arabs were in possession of a good numerical notation, and of Brahmagupta's work on arithmetic and algebra; while before the end of the ninth century they were acquainted with the masterpieces of Greek mathematics in geometry, mechanics, and astronomy. I have now to explain what use they made of these materials.

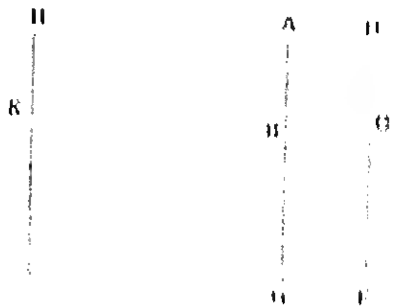
The first and in some respects the most illustrious of the Arabian mathematicians was *Mohammed ibn Musa Abu Djefar* **Al-Khwârizmî**. There is no common agreement as to which of these names is the one by which he is to be known: the last of them refers to the place where he was born or in connection with which he was best known, and I am told that it is the one by which he would have been usually known among his contemporaries. I shall therefore refer to him by that name; and shall also generally adopt the corresponding titles to designate the other Arabian mathematicians. Until recently this was almost always written in the corrupt form **Alkarismi**,

and though this way of spelling it is incorrect, it has been sanctioned by so many writers that I shall make use of it. We know nothing of Alkarismi's life except that he was a native of Khorassan, and librarian of the caliph Al Mamun; and that he accompanied a mission to Afghanistan, and possibly came back through India. On his return, about 830, he wrote an algebra which is founded on that of Brahmagupta, but in which some of the proofs rest on the Greek method of representing numbers by lines: it was published by Rosen, with an English translation, at London in 1831. Besides this algebra Alkarismi wrote an astronomy, and a treatise on arithmetic. An anonymous tract termed *Algoritmi de numero Indorum* which is in the university library at Cambridge is believed to be a Latin translation of the latter; it was published by B. Boncompagni at Rome in 1857.

The algebra of Alkarismi holds a most important place in the history of mathematics, for we may say that the subsequent Arabian and all the early medieval works on algebra were founded on it. The work is termed *Al-gebr we' l mukabala*: *al-gebr*, from which the word algebra is derived, means that any the same magnitude may be added to or subtracted from both sides of an equation; *al mukabala* means the combination of like terms into a single term. The unknown quantity is termed either "the thing" or "the root" (i.e. of a plant) and from the latter phrase our use of the word root as applied to the solution of an equation is derived. The square of the unknown is called "the power." All the known quantities are numbers.

The work is divided into five parts. In the first Alkarismi gives, without any proofs, the rules for the solution of quadratic equations, which he divides into six classes of the forms $ax^2 = bx$, $ax^2 = c$, $bx = c$, $ax^2 + bx = c$, $ax^2 + c = bx$, and $ax^2 = bx + c$. He only considers real and positive roots, but he recognizes the existence of two roots, which as far as we know was never done by the Greeks. It is somewhat curious that when both roots are positive he generally only takes that root which is

derived from the negative value of the radical. He next gives geometrical proofs of these rules in a manner analogous to that of Euclid II. 4. For example to solve the equation $x^2+10x=39$, or any equation of the form $x^2+px=q$, he gives two methods of which one is as follows. Let AB represent the value of x, and construct on it the square $ABCD$. Produce DA and DC to H and F so that $AH=CF=5$ (or $\frac{1}{2}p$); and complete the figure as drawn below. Then the areas AC, HB, and BF represent the magnitudes x^2, $5x$, and $5x$. Thus the left-hand



side of the equation is represented by the sum of the areas AC, HB, and BF, that is by the gnomon HCG. To both sides of the equation add the square KG, the area of which is 25 (or $\frac{1}{4}p^2$), and we shall get a new square whose area is by hypothesis equal to $39+25$, i.e. to 64 (or $q+\frac{1}{4}p^2$) and whose side therefore is 8. The side of this square DH which is equal to 8 will exceed HA which is equal to 5 by the value of the unknown required, which is therefore 3. In the third part of the book Alkarismi considers the product of $(x \pm a)$ and $(x \pm b)$. In the fourth part he states the rules for addition and subtraction of expressions which involve the unknown, its square, or its square root; gives rules for the calculation of square roots; and concludes with the theorems that $a\sqrt{b}=\sqrt{a^2b}$ and $\sqrt{a}\sqrt{b}=\sqrt{ab}$. In the fifth and last part he gives some problems, such for example as to find two numbers whose sum is 10 and the difference of whose squares is 40.

ll these early works there is no clear distinction between
tic and algebra, and we find the account and explana-
arithmetical processes mixed up with algebra and treated
of it. It was from this book then that the Italians
ained not only the ideas of algebra but also of an arith-
ounded on the decimal system. This arithmetic was
own as *algorism*, or the art of Alkarismi, which served
nguish it from the arithmetic of Boethius; and this
emained in use till the eighteenth century.

work commenced by Alkarismi was carried on by
bn Korra (see p. 141) born at Harran in 836 and died
who was one of the most brilliant and accomplished
produced by the Arabs. He issued translations of
f works of Euclid, Apollonius, Archimedes, and Ptolemy.
wrote several original works. All of these are lost with
eption of a fragment of one on algebra, which was found
National Library in Paris by M. Sedillot; this consists
chapter on cubic equations, which are solved by the aid
etry in somewhat the same way as that given later (see
).

subsequent development of algebra seems to have been
apid, but it remained entirely rhetorical. The problems
hich the Arabs were concerned were either the solution
ations, problems leading to equations, or properties of
rs. In the solution of equations they were successful
leterminate problems, but I do not know of any case in
an indeterminate problem was solved generally. In the
of numbers they discovered expressions for the sum of
st, second, third, and fourth powers of the first n natural
rs. **Alkhodjandi**, who was alive in 992, stated that it
npossible to solve the equation $x^3+y^3=z^3$ in positive
rs, or in other words that the sum of two cubes can
be a cube. Whether the proof he gave was accurate,
ether, as is more likely, it was the result of a wide in-
n, it is now impossible to say; but the fact that he
ated such a theorem will serve to illustrate the extra-

ordinary progress they had made far better than a mere list of propositions.

Some minor improvements in notation were introduced, such e.g. as the introduction of a line to separate the numerator from the denominator of a fraction. Hence a line between two symbols came to be used as a symbol of division (see p. 213). **Alhossein** (980–1037) invented the rule for testing the results of addition by "casting out the nines"; and wrote a treatise founded on that of **Diophantus** on rational right-angled triangles. The most prominent algebraist of a later date was **Alkarki** (circ. 1000) whose work on algebra, containing the general solution of a cubic equation, was published by F. Woepcke at Paris in 1853, and whose treatise on arithmetic was translated into German in 1878 by Hochheim.

Even where the methods of Arab algebra are quite general the applications are confined in all cases to numerical problems, and the algebra is so arithmetical that it is difficult to treat the subjects apart. From their books on arithmetic and from the observations scattered through various works on algebra we may say that the methods used by the Arabs for the four fundamental processes were analogous to, but more cumbrous than, those now in use (see chapter XI.); but the problems to which the subject was applied were similar to those given in modern books, and were solved by similar methods, such as rule of three, &c.

I am not concerned with the Arabian views of *astronomy* or the value of their observations, but I may just remark in passing that the Arabs accepted the theory as laid down by Hipparchus and **Ptolemy**, and did not materially alter or advance it.

Like the Greeks the Arabs never used *trigonometry* except in connection with astronomy: but they introduced the trigonometrical expressions which are now current. These seem to have been the invention of **Albategni**, born at Batan in 877 and died at Bagdad in 929, who was among the earliest of the many distinguished Arabian astronomers. He wrote *The science of the stars* (published by Regiomontanus at Nuremberg

in 1537) and in it he determined his angles by "the semi-chord of twice the angle," i.e. by the *sine* of the angle (taking the radius vector as unity). Hipparchus and Ptolemy, it will be remembered, had used the chord. Albategni seems to have been ignorant of the previous introduction of sines by Arya-Bhatta and Brahmagupta. Shortly after his death **Albuzjani** who is better known as **Abul-Wafa** (940–998) introduced all the trigonometrical functions, and constructed tables of tangents and cotangents. He was celebrated not only as an astronomer but as one of the most distinguished geometricians of his time.

The Arabs were at first content to take Euclid and Apollonius for their text-books in *geometry* without attempting to comment on them, but **Alhazen** (born at Bassora in 987 and died at Cairo in 1038) issued in 1036 a collection of problems something like the *Data* of Euclid, which was translated by Sedillot and published at Paris in 1836. Besides commentaries on the definitions of Euclid and on the *Almagest* he also wrote an *Optics* which shews that he was a geometrician of considerable power: this was published at Bâle in 1572, and served as the foundation for Kepler's treatise. In it he gives, amongst other things, a geometrical solution of the problem to find at what point of a concave mirror a ray from a given point must be incident so as to be reflected to another given point. Another geometrician of a slightly later date was **Abd-al-gehl** (circ. 1100) who wrote on *conic sections*, and was also the author of three small geometrical tracts.

It was shortly after the last of the mathematicians mentioned above that Bhaskara, the third great Hindoo mathematician, flourished: there is every reason to believe that he was familiar with the works of the Arab school as described above, and also that his writings were at once known in Arabia.

The Arab schools continued to flourish to the fifteenth century. But they produced no other mathematician of any exceptional genius, nor was there any great advance on the methods indicated above, and it is unnecessary for me to crowd

my pages with the names of a number of writers who did not materially affect the progress of the science in Europe.

I have not alluded to a strange theory which has been accepted by many writers, but which seems to me to be most improbable. According to this theory there were two rival schools of thought in Arabia, one of which derived its mathematics entirely from Greek sources and represented numbers by lines, and the other from Hindoo sources and represented numbers by abstract symbols—each disdaining to make any use of the authorities preferred by its rival.

From this rapid sketch it will be seen that the work of the Arabs in arithmetic algebra and trigonometry was of a high order of excellence. They appreciated geometry and the applications of geometry to astronomy, but they did not extend the bounds of the science. It may also be added that they made no special progress in statics or optics or hydrostatics, though there is abundant evidence that they had a thorough knowledge of practical hydraulics.

The general impression left on my mind is that the Arabs were quick to appreciate the work of others—notably of the Greek masters and of the two Hindoos who produced original work—but like the ancient Chinese and Egyptians they were unable to systematically develope a subject to any considerable extent. Their schools may be taken to have lasted in all for about 650 years, and if the work produced be compared with that of Greek or modern European writers it is as a whole second-rate both in quantity and quality.

CHAPTER X.

THE INTRODUCTION OF ARABIAN MATHEMATICAL WORKS INTO EUROPE.

In the last chapter but one I discussed the development of European mathematics to a date which corresponds roughly with the end of the "dark ages"; and in the last chapter I traced the history of the mathematics of the Hindoos and Arabs to the same date. The two or three centuries that follow and form the subject of this chapter are characterized by the introduction of the Arabian mathematical text-books and of Greek books derived from Arabian sources, and the assimilation of the new ideas thus presented.

It was however from Spain and not from Arabia that Arabian mathematics came into western Europe. The Moors had established their rule in Spain in 747, and by the tenth or eleventh century had attained a high degree of civilization. Though their political relations with the caliphs at Bagdad were somewhat unfriendly, they gave a ready welcome to the works of the great Arabian mathematicians. In this way the Arab translations of Euclid, Archimedes, Ptolemy, and perhaps of other Greek writers, together with the works of the Arabian algebraists, were read and commented on at the three great Moorish universities or schools of Granada, Cordova, and Seville. It seems probable that these works represent the extent of Moorish learning, but as all knowledge was jealously guarded from any Christians, it is impossible to speak with certainty either on this point or on that of the time when the Arab books were first introduced into Spain.

The earliest Moorish writer of whom I can find any men-

tion is **Geber ibn Aphla**, who was born at Seville and died towards the latter part of the eleventh century at Cordova. His works which deal chiefly with astronomy and trigonometry were translated into Latin by Gerard and published at Nuremberg in 1533. He seems to have discovered the theorem that the sines of the angles of a spherical triangle are proportional to the sines of the opposite sides.

Another Arab of about the same date was **Arzachel***, who was living at Toledo in 1080. He suggested that the planets moved in ellipses, but his contemporaries with scientific intolerance declined to argue about a statement which was contrary to that made by Ptolemy in the *Almagest*.

During the course of the twelfth century copies of the books used in Spain were obtained in western Christendom. The first step towards procuring a knowledge of Arab and Moorish science was taken by an English monk **Adelhard** of Bath, who, disguised as a Mohammedan student, got into Cordova about 1120 and obtained a copy of Euclid's *Elements*. This copy translated into Latin was the foundation of all the editions known in Europe till 1533. How rapidly a knowledge of the work spread we may judge when we recollect that before the end of the thirteenth century Roger Bacon was familiar with it, while before the close of the fourteenth century the first five books formed part of the regular curriculum at some, if not all, universities. Adelhard also procured a copy of or commentary on Alkarismi's work, which he likewise translated into Latin.

During the same century other translations of the Arab text-books or commentaries on them were obtained. Amongst those who were most influential in introducing Moorish learning into Europe I must mention **Abraham Ben Ezra**. Ben Ezra, who was born at Toledo in 1097 and died at Rome in 1167, was one of the most distinguished Jewish rabbis who had settled in Spain, for it must be recollected that the Jews

* See his life by Bahli, circ. 1600, reprinted by Boncompagni in the *Bulletino di Bibliografia* for 1872.

were tolerated and even protected by the Moors on account of their medical skill. Besides some astronomical tables and an astrology, Ben Ezra wrote an arithmetic, a short analysis of which was published by O. Terquem in Liouville's Journal for 1841. In this he explains the Arab system of numeration with nine symbols and a zero, gives the fundamental processes of arithmetic, and the rule of three.

Another European who was induced by the reputation of the Arab schools to go to Toledo was **Gerard*** who was born at Cremona in 1114 and died in 1187. He translated the Arab edition of the *Almagest*, the works of Alhazen, and the works of Alfarabius, another Arab whose name is otherwise unknown to us. There can be no doubt that the Arabic numerals were introduced into Spain together with the Arabian text-books, but this translation of Ptolemy which was made in 1136 contains the earliest use of them to which we can definitely point. Gerard also wrote a short treatise on algorism which exists in manuscript in the Bodleian Library at Oxford.

Among the contemporaries of Gerard was **John Hispalensis**, who was originally a rabbi, but was converted to Christianity and baptized under the name given above. He made translations of several Arab and Moorish works, and also wrote an algorism which contains the earliest examples of the extraction of the square roots of numbers by the aid of decimals.

The thirteenth century is distinguished by the names of Leonardo of Pisa, and Roger Bacon, the Franciscan monk of Oxford.

Leonardo *Fibonacci* (i.e. filius Bonacii) was born at Pisa in 1175. His father Bonacci was a merchant and was sent by his fellow-townsmen to control the custom-house at Bugia in Barbary. There Leonardo was educated, and became acquainted with the Arabic system of numeration and with the great Arab work on algebra by **Alkarismi** which was described in the last chapter. It would seem that Leonardo was en-

* See Boncompagni's *Della vita e delle opere di Gherardo Cremonese*, Rome, 1851.

trusted with some duties in connection with the custom-house which required him to travel. He returned to Italy about 1200, and in 1202 published a work called *Algebra et almuchabala* (the title being taken from Alkarismi's work) but generally known as the *Liber abbaci*. He there explains the Arabic system of numeration, and remarks on its great advantages over the Roman system. He then gives an account of algebra, and points out the convenience of using geometry to get rigid demonstrations of algebraical formulæ. He solves a few quadratic equations, and states some methods for the solution of indeterminate equations; but all the algebra is rhetorical.

This work is especially interesting since it had a wide circulation and practically introduced the use of the Arabic numerals into Christian Europe. The language of Leonardo implies that they were previously unknown to his countrymen; he says that having had to spend some years in Barbary he there learnt the Arabic system which he found much more convenient than that used in Europe; he therefore published it "in order that the Latin* race might no longer be deficient in that knowledge." Now Leonardo was very widely read, and had travelled in Greece, Sicily, and Italy: and there is therefore every presumption that the system was not then in general use in Europe. Within another thirty or forty years it was employed by merchants in Italy by the side of the old system. Though Leonardo introduced it into commercial affairs, it is probable that a knowledge of it as a method which was current in the East was previously not uncommon, for the intercourse between Christians and Mohammedans was sufficiently close for each to learn something of the language and common practices of the other. We can also hardly suppose that the Italian merchants were ignorant of the method of keeping accounts used by some of their best customers; and we must recollect too that there were numerous Christians

* Dean Peacock says that the earliest known use of the word Italians to describe the inhabitants of Italy occurs about the middle of the thirteenth century.

who had escaped or been ransomed after serving the Mohammedans as slaves.

The majority of mathematicians must have already known of the system from the works of Ben Ezra, Gerard, and John Hispalensis. But shortly after the appearance of Leonardo's book Alphonso of Castile (in 1252) published some astronomical tables founded on observations made in Arabia, which were computed by Arabs, and which were expressed in Arabic notation. Alphonso's tables had a wide circulation among men of science and were largely instrumental in bringing these numerals into universal use among mathematicians. By the end of the thirteenth century it was generally assumed that all scientific men would be acquainted with the system: thus Roger Bacon writing in that century recommends the algorism (i.e. the arithmetic founded on the Arab notation) as a necessary study for theologians who ought he says "to abound in the power of numbering." We may then consider that by the year 1300, or at the latest 1350, these numerals were familiar both to mathematicians and to merchants.

So great was Leonardo's reputation that the emperor Frederick II. stopped at Pisa in 1225 in order to hold a sort of mathematical tournament to test Leonardo's skill of which he had heard such marvellous accounts. This is the first time that we meet with an instance of those challenges to solve particular problems which were so common in the sixteenth and seventeenth centuries. The first question propounded to Leonardo was to find a number of which the square when either increased or decreased by 5 would remain a square. He gave an answer, which is correct, namely 41/12. The next question was to find by the methods used in the tenth book of Euclid a line whose length x should satisfy the equation $x^3+2x^2+10x=20$. Leonardo showed by geometry that the problem was impossible, but he gave an approximate value of the root of this equation. An analysis of his method was published in Liouville's *Journal* for 1855. The third and last question was as follows. Three men A, B, C, possess a sum of money u, their shares being in

the ratio 3 : 2 : 1. A takes away x, keeps half of it, and deposits the remainder with D; B takes away y, keeps two-thirds of it, and deposits the remainder with D; C takes away all that is left, namely z, keeps five-sixths of it, and deposits the remainder with D. This deposit with D is found to belong to A, B, and C in equal proportions. Find u, x, y, and z. Leonardo shewed that the problem was indeterminate and gave as one solution $u=47$, $x=33$, $y=13$, $z=1$. His opponents failed to solve any of these questions.

The chief work of Leonardo is the *Liber abbaci* alluded to above. He also wrote a geometry termed *Practica geometriae* in which among other propositions and examples he finds the area of a triangle in terms of its sides: this was issued in 1220. He subsequently published a *Liber quadratorum* dealing with problems similar to the first of the questions propounded at the tournament, and a tract dealing with determinate algebraical problems. The latter are all solved by the rule of false assumption in the manner explained on p. 95.

His works have been published in 2 volumes under the title *Scritti di Leonardo Pisano* by B. Boncompagni at Rome in 1857 and 1862.

The emperor **Frederick II.** who was born in 1194, succeeded to the throne in 1210, and died in 1250, was not only interested in science, but did more than any other single man of this century to disseminate a knowledge of the works of the Arab mathematicians in northern and western Europe. I have already mentioned that the presence of the Jews had been tolerated in Spain on account of their medical skill and scientific knowledge: and as a matter of fact the titles of physician and algebraist* were for a long time nearly synonymous. The Jewish physicians were thus admirably fitted both to get copies of the Arab works, and to translate them. Frederick II. made use of this fact to engage a staff of learned Jews to translate the Arab works which he obtained, though there is

* Thus the reader may recollect that when in *Don Quixote* Carasco is wounded an *algebrista* is summoned to bind up his wounds.

no doubt that he gave his patronage to them the more readily because it was singularly offensive to the pope with whom he was then engaged in a quarrel. At any rate by the end of the thirteenth century copies of Euclid, Archimedes, Apollonius, Ptolemy, and some of the Arab works on algebra were obtainable from this source, and by the end of the next century were not uncommon. From this time then we may say that the development of learning in western Europe was independent of the aid of the Arabian schools.

The only mathematician of this century who can rank with Leonardo is *Roger* Bacon*, who was born near Ilchester in 1214 and died at Oxford on June 11, 1294. He was the son of royalists, most of whose property had been confiscated at the end of the civil wars. At an early age he was entered as a student at Oxford, and is said to have taken orders in 1233. In 1234 he removed to Paris, then the intellectual capital of western Europe, where he lived for some years devoting himself especially to languages and physics; and there he spent on books and experiments all that remained of his family property and his savings. He returned to Oxford at some time between 1240 and 1250, and occupied himself in teaching science. For some years he laboured incessantly; his lecture room was crowded, but all that he earned was spent in buying manuscripts and instruments. He tells us that altogether at Paris and Oxford he had spent over £2000 in this way—a sum which represents at least £20,000 now-a-days.

Bacon strove hard to replace logic in the university curriculum by mathematical and linguistic studies, but the influences of the age were still too strong for him. His glowing eulogy on "divine mathematics" which should form the foundation of a liberal education, and which "alone can purge the intellect and fit the student for the acquirement of all knowledge" fell on deaf ears. We can judge how small was the amount of

* See *Roger Bacon, sa vie, ses ouvrages...* by E. Charles, Paris, 1861; and *Roger Bacon, eine Monographie*, by Schneider, Augsburg, 1873. The first of these is very eulogistic, the latter somewhat severely critical.

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Bacon strove hard to replace logic in the university curriculum by mathematical and linguistic studies, but the influences of the age were still too strong for him. His glowing eulogy on "divine mathematics" which should form the foundation of a liberal education, and which "alone can purge the intellect and fit the student for the acquirement of all knowledge" fell on deaf ears. We can judge how small was the amount of

* See *Roger Bacon, sa vie, ses ouvrages...* by E. Charles, Paris, 1861; and *Roger Bacon, eine Monographie*, by Schneider, Augsburg, 1873. The first of these is very eulogistic, the latter somewhat severely critical.

geometry which was implied in the quadrivium, when he tells us that few students at Oxford got beyond Euc. I. 5; though we might perhaps have inferred as much from the character of the work of Boethius.

At last worn out, neglected, and ruined Bacon was persuaded by his friend Grosseteste, the great Bishop of Lincoln, to renounce the world and take the Franciscan vows. The society to which he now found himself confined was singularly uncongenial to him, and he beguiled the time by writing on scientific questions, and perhaps lecturing. The superior of the order heard of this, and in 1257 forbad him to lecture or publish anything under penalty of the most severe punishments, and at the same time directed him to take up his residence at Paris where he could be more closely watched. Clement IV. when in England had heard of his abilities, and in 1266 when he became pope he invited Bacon to write. The Franciscan order reluctantly permitted him to do so, but they refused him any assistance. With great difficulty Bacon obtained sufficient money to get paper and the loan of books, and within the short space of fifteen months he produced in 1267 his *Opus majus* with two supplements which summarized all that was then known in science, and laid down the principles on which not only science, but philosophy and literature, should be studied. He stated as the fundamental principle that the study of natural science must rest solely on experiment; and in the fourth part he explains in detail how all sciences rest ultimately on mathematics, and progress only when their fundamental principles are expressed in a mathematical form. Mathematics, he says, should be regarded as the alphabet of all philosophy.

The results that he arrives at in this and his other works are nearly in accordance with modern ideas, but were too far in advance of that age to be capable of appreciation or perhaps even of comprehension, and it was left for later generations to rediscover his works, and give him that credit which he never experienced in his lifetime. In astronomy he laid down the

principles for a reform of the calendar, explained the phenomenon of shooting stars, and stated that the Ptolemaic system was unscientific in so far as it rested on the assumption that circular motion was the natural motion of a planet, while the complexity of the explanations required made it improbable that the theory was true. In optics he enunciated the laws of reflexion and in a general way of refraction of light; and used them to give a rough explanation of the rainbow and of magnifying glasses. Most of his experiments in chemistry were directed to the transmutation of metals, and led to no results. He gave the composition of gunpowder; but there is no doubt that it was not his own invention, though it is the earliest European mention of it. On the other hand some of his results in these subjects appear to be guesses which are more or less ingenious, while certain statements he makes are certainly erroneous.

He wrote numerous works which developed in detail the principles laid down in his *Opus majus* in the years immediately following its publication. Most of these have now been published, but I do not know of the existence of any complete edition. They deal only with applied mathematics and physics.

Clement took no notice of the great work for which he had asked, except to obtain leave for Bacon to return to England. On the death of Clement, the general of the Franciscan order was elected pope, and took the title of Nicholas IV. Bacon's investigations had never been approved of by his superiors, and he was now ordered to return to Paris, where we are told he was immediately accused of magic; he was condemned in 1280 to imprisonment for life, and was only released about a year before his death.

Amongst the minor mathematicians of this century was *Giovanni* Campanus who translated Euclid's *Elements*, and wrote a commentary thereon in which he discussed the properties of a regular re-entrant pentagon. Besides some minor works he wrote *The Theory of the Planets* which was a free translation of the *Almagest*. Another mathematician of about

the same time was Jordanus, who composed text-books on arithmetic, algebra, maps, and astronomy.

The history of the fourteenth century like that of the one preceding it is mostly concerned with the introduction and assimilation of the Arabian mathematical text-books and the Greek books derived from Arabian sources. There was contemporaneously a revolt of the universities against the intellectual tyranny of the schoolmen. This was largely due to Petrarch, who to his own generation was celebrated as a humanist rather than as a poet, and who exerted all his power to destroy scholasticism, and encourage scholarship. The result of these influences on the study of mathematics may be seen in the regulations made at the university of Vienna in 1389 as to the manner in which the last three out of the seven-years course for a master's degree should be spent. Before a student could take that degree he was expected to have mastered "five books of Euclid, common perspective, proportional parts, the measurement of superficies, and the *Theory of the Planets*." The book last named is the treatise by Campanus. Similar rules existed at Prague and Leipzig. In 1366 a reform to the same effect had been made at Paris, and a couple of years later at Oxford and Cambridge; but unfortunately the text-books required at these universities are not mentioned; still it seems reasonable to suppose that the standard required was about the same as that at Vienna, which was itself an off-shoot of Paris. This was a fairly respectable mathematical standard, but I would remind the reader that there was no such thing as "plucking" in a mediæval university. The student had to keep an act or give a lecture on certain subjects, but whether he did it well or badly he got his degree, and it is probable that it was only the few students whose interests were mathematical who really mastered the subjects mentioned above.

By the middle of the fifteenth century printing was invented and the facilities it gave for disseminating knowledge were so great as to revolutionize the progress of science. We

have arrived at a time when the results of Arab and Greek science were known in Europe; and this perhaps then is as good a date as can be fixed for the close of this period and the commencement of that of the renaissance. The mathematical history of the renaissance begins with the career of Regiomontanus; but before proceeding with the general history it will be convenient to collect together the chief facts connected with the subsequent development of arithmetic to the year 1637. To this the next chapter is devoted.

CHAPTER XI.

THE DEVELOPMENT OF ARITHMETIC TO THE YEAR 1637*.

We have seen in the last chapter that by the end of the thirteenth century the Arabic arithmetic had been fairly introduced into Europe and was practised by the side of the older arithmetic which was founded on the work of Boethius. It will be convenient to depart from the chronological arrangement and briefly to sum up the subsequent history of arithmetic, but I hope by references in the next chapter to the inventions and improvements in arithmetic here described that I shall be able to keep the order of events and discoveries quite clear.

The older arithmetic consisted of two parts: practical arithmetic or the art of calculation which was taught by means of the abacus and possibly the multiplication table, and theoretical arithmetic, by which was meant the ratios and properties of numbers taught according to Boethius—a knowledge of the latter being confined to professed mathematicians. The theoretical part of this system continued to be taught till the middle of the fifteenth century; and the practical part of it was used by the smaller tradesmen in England†, Germany, and France till the beginning of the seventeenth century. Any one who cares to see how the abacus can be used for

* See the article on Arithmetic by Dean Peacock in the *Encyclopaedia Metropolitana*, Vol. I., London, 1845; and *Arithmetical Books* by A. de Morgan, London, 1847.

† See e.g. Chaucer, *The Miller's Tale*, v. 22—25; Shakespeare, *The Winter's Tale*, Act IV. Sc. 3; *Othello*, Act I. Sc. 1. I am not sufficiently familiar with early French or German literature to know whether they contain any references to the use of the abacus. I believe that the Exchequer division of the High Court of Justice derives its name from

multiplication, division, and even complicated sums will find the rules together with examples on them in the *Arithmetike* by R. Record, London, 1st ed. 1540.

The new Arabian arithmetic was called *algorism* or the art of Alkarismi to distinguish it from the old or Boethian arithmetic. The text books on algorism commenced with the Arabic system of notation, and began by giving rules for addition, subtraction, multiplication, and division; the principles of proportion were then applied to various practical problems, and the books usually concluded with algebraic formulae for most of the common problems of commerce. Algorism was in fact a mercantile arithmetic, though at first it also included all that was then known as algebra. Thus algebra has its origin in arithmetic; and to most people the term *universal arithmetic* by which it was sometimes designated conveys a far more accurate impression of its objects and methods than the more elaborate definitions of modern mathematicians — certainly better than the definition of Sir William Hamilton as the science of pure time, or that of Prof. de Morgan as the calculus of succession. No doubt logically there is a marked distinction between arithmetic and algebra, for the former is the theory of discrete magnitude while the latter is that of continuous magnitude; but a scientific distinction such as this is of quite recent origin, and the idea of continuity was not introduced into mathematics before the time of Kepler. Of course the fundamental rules of this algorism were not at first strictly proved (that is the work of advanced thought), but until the time of Cocker 1677 there was some discussion of the principles involved; since then very few arithmeticians have attempted to justify or prove the processes used, or to do more than enunciate rules and illustrate their use by a few numerical examples.

The trade of Europe during the thirteenth and fourteenth centuries was mostly in Italian hands; and the obvious advan-

the table before which the judges and officers of the court originally sat: this was covered with black cloth divided into squares or chequers by white lines, and was apparently used as an abacus.

tages of the algoristic system for mercantile purposes led to its general adoption in Italy. The rapid spread of the use of Arabic numerals and arithmetic through the rest of Europe seems to have been quite as largely due to the makers of almanacks and calendars as to merchants and men of science. Perhaps the oriental origin of the symbols gave them an attractive flavour of magic, but there seem to have been very few calendars after the year 1300 in which an explanation of the system was not included. Towards the middle of the fourteenth century the rules of arithmetic *de algorismo* were also added, and by the year 1400 we may consider that they were generally known throughout Europe, and were used in most scientific and astronomical works. Most merchants continued however to keep their accounts in Roman numerals till about 1550, and monasteries and colleges till about 1650: though in both cases it is probable that the processes of arithmetic were performed in the algoristic manner. The Arabic numerals were introduced into Constantinople by Planudes at about the same time as into Italy (see p. 111).

The history of algorism in Europe begins then with its use by Italian merchants: and it is especially to the Florentine traders and writers that we owe its early development and improvement. It was they who invented the system of book-keeping by double entry. In this system every transaction is entered on the credit side in one ledger, and on the debtor side in another; e.g. if cloth is sold to *A*, *A*'s account is debited with the price, and the stock book containing the transactions in cloth is credited with the amount sold. It was they too who arranged the problems to which arithmetic could be applied in different classes, such as rule of three, interest, profit and loss, &c. They also reduced the fundamental operations of arithmetic "to seven*, in reverence" says Pacioli "of the seven gifts

* Brahmagupta had enumerated twenty processes besides eight subsidiary ones; and stated that "a distinct and several knowledge of these" was "essential to all who wished to be calculators."

of the Holy Spirit: namely, numeration, addition, subtraction, multiplication, division, raising to powers, and extraction of roots": and whatever we may think of Pacioli's reason for this classification of the fundamental processes, the result was satisfactory.

The processes of algoristic arithmetic were at first very cumbersome, and their subsequent simplification was chiefly due to the Italians and English. It is curious that the English should have played so large a part in developing arithmetic, but from the first they showed great aptitude for commercial arithmetic and algebra, while as they had never generally adopted the Boethian system, they were not hampered by having to get rid of it; and so in spite of the small attention paid to mathematics in general they were among the most expert arithmeticians of the fifteenth and sixteenth centuries. We may perhaps say that the study of arithmetic in England and Italy was more than half a century in advance of that on the rest of the continent.

The chief improvements introduced into the early Italian algorism were (i) the simplification of the four fundamental processes; (ii) the introduction of signs for plus, minus, and equality; (iii) the invention of logarithms; and (iv) the use of decimals. I will take these in succession.

(i) In addition and subtraction the Arabs usually worked from left to right. The modern plan of working from right to left is shorter and was introduced by an Englishman named Garth. The old plan continued in partial use till about 1600; it would even now be more convenient in approximations where it is only necessary to keep a certain number of places of decimals.

The Indians and Arabs had several systems of multiplication. These were all somewhat laborious, and were made the more so as multiplication tables were unknown or at any rate unused. The operation was regarded as one of considerable difficulty, and the test of the accuracy of the result by "casting out the nines" was invented by the Arabs as a check

on the correctness of their work. Various other systems of multiplication were subsequently employed in Italy, of which several examples are given by Pacioli and Tartaglia; and the use of the multiplication table—at least as far as 5×5, from which the result of multiplications for all numbers up to 10×10 can be deduced*—became common. The system of multiplication now in use seems to have been first introduced at Florence. The difficulty which all but professed mathematicians experienced in multiplying led to the invention of several mechanical ways of effecting the process. Of these the most celebrated is that of Napier's rods invented in 1617 a full description of which will be found in Peacock's article on arithmetic.

If multiplication was considered difficult, division was at first regarded as a feat which could only be performed by skilled mathematicians. The present system was in use in Italy as early as the beginning of the fourteenth century, but it was not till the beginning of the eighteenth century that it was universally adopted in the rest of Europe. Till then the method generally employed was that known as the *galley* or *scratch* system. This was used as late as 1798 in Bernouilli's translation into French of Euler's *Anleitung zur Arithmetic.* The following example from Tartaglia will serve to illustrate the method: the numbers in thin type are supposed to be scratched out in the course of the work.

To divide 1330 by 84

$$\begin{array}{llllll}
 & 0 & 7 & & & \\
 & 4 & 9 & & & \\
0 & 5 & 9 & 0 & & \\
1 & 3 & 3 & 0 & (15 & \\
 & 8 & 4 & 4 & & \\
 & & 8 & & &
\end{array}$$

The process is as follows. First write the 84 below the

* The rule was called the *regula ignavi*, and is a statement of the identity

$$(5+a)(5+b) \equiv (5-a)(5-b) + 10(a+b).$$

1330, as in the work, then 84 will go once into 133, hence the first figure in the quotient is 1. Now 1 × 8 = 8, which subtracted from 13 leaves 5. Cancel out the 13 and the 8, and we have as the result of the first step

$$\begin{array}{ccccl} & 5 & & & \\ 1 & 3 & 3 & 0 & (\,1 \\ & 8 & 4 & & \end{array}$$

Next 1 × 4 = 4, which subtracted from 53 leaves 49. Cancel out the 53 and the 4, and we have as the next step

$$\begin{array}{ccccl} & 4 & & & \\ & 5 & 9 & & \\ 1 & 3 & 3 & 0 & (\,1 \\ & 8 & 4 & & \end{array}$$

which shows a remainder 490.

We have now to divide 490 by 84. The next figure in the quotient will therefore be 5, and re-writing the divisor we have

$$\begin{array}{ccccl} & 4 & & & \\ & 5 & 9 & & \\ 1 & 3 & 3 & 0 & (\,15 \\ & 8 & 4 & 4 & \\ & & 8 & & \end{array}$$

Then 5 × 8 = 40, and this subtracted from 49 leaves 9. Cancel the 49 and the 8, and we have the following result.

$$\begin{array}{ccccl} & 4 & 9 & & \\ & 5 & 9 & & \\ 1 & 3 & 3 & 0 & (\,15 \\ & 8 & 4 & 4 & \\ & & 8 & & \end{array}$$

Next 5 × 4 = 20, and this subtracted from 90 leaves 70. Cancel the 90 and the 4, and the final result shewing a remainder 70 is

$$\begin{array}{ccccl} & & 7 & & \\ & 4 & 9 & & \\ & 5 & 9 & 0 & \\ 1 & 3 & 3 & 0 & (\,15 \\ & 8 & 4 & 4 & \\ & & 8 & & \end{array}$$

The three extra steps inserted in Tartaglia's work are un-

necessary, but they do not affect the work, as it is evident that a figure in the dividend may be shifted one or more places up in the same vertical column if it is convenient to do so.

The mediæval writers were acquainted with the method now in use, but considered the scratch method more simple. In some cases the latter is very clumsy as may be illustrated by the following example taken from Pacioli. The object is to divide 23400 by 100. The result is obtained thus

```
      0
    0 4 0
  0 3 4 0 0
  2 3 4 0 0 ( 234
  1 0 0 0 0
    1 0 0
      1
```

(ii) The signs + and − to represent addition and subtraction occur in Widman's arithmetic published in 1489 (see p. 185), but were first brought into general notice, at any rate as symbols of operation, by Stifel in 1544 (see p. 192). The sign = to denote equality was introduced by Recorde in 1540 (see p. 191). I believe I am correct in saying that Vieta in 1591 was the first well-known writer who used these signs consistently throughout his work, and it was not until the beginning of the seventeenth century that they were recognized and well-known symbols.

(iii) The invention of logarithms, without which many of the numerical calculations which have constantly to be made would be practically impossible, was due to *John* Napier* of Merchistoun who was born in 1550 and died on April 3, 1617. Napier spent most of his life on the family estate near Edinburgh, and took an active part in the political and religious controversies of the day. The business of his life was to shew that the pope was antichrist, but his favorite amusement was the study of mathematics and science. As soon as the use of exponents became common in algebra the introduction of logarithms would naturally follow, but Napier reasoned out the

* See the *Memoirs of Napier* by Mark Napier, Edinburgh, 1834.

result without the use of any symbolic notation to assist him, and the invention of logarithms was so far from being a sudden inspiration that it was the result of many years' efforts with a view to abbreviate the processes of multiplication and division. The first announcement of the discovery was made in his *Mirifici logarithmorum canonis descriptio* published in 1614, and of which an English translation was issued in the following year. This work explained the nature of logarithms by a comparison between corresponding terms of an arithmetical and geometrical progression. It illustrated their use, and gave tables of the logarithms of the sines and tangents of all angles for differences of every minute, calculated to seven places of decimals. The logarithm of a quantity n was what we should now express by the formula $10^7 \log (10^7/n)$. This work is the more interesting to us as it is the first valuable contribution to the progress of mathematics which was made by any British writer.

The method by which the logarithms were calculated was explained in the *Constructio* a posthumous work issued in 1619; it seems to have been very laborious and depended on forming an immense number of geometrical means of various numbers, and not on finding the approximate value of a convergent series. Napier had determined to change the base to one which was a power of 10, but died before he could effect it.

The rapidity with which the use of logarithms was adopted in England and elsewhere was largely due to Briggs; while among the most prominent of those who subsequently helped to introduce their use on the continent was Kepler.

Henry **Briggs*** was born near Halifax in 1556. He was educated at St John's College, Cambridge, took his degree in 1581, and obtained a fellowship in 1588. He was elected to the Gresham professorship of geometry in 1596, and in 1619 became Savilian professor at Oxford, a chair which he held until his death on Jan. 26, 1630.

* See *The lives of the Professors of Gresham College* by J. Ward, London, 1740.

Briggs was amongst the earliest to recognize the value of Napier's invention; but he deemed the base to which Napier's logarithms were calculated to be very inconvenient. He accordingly visited Napier in 1616, and the change to a decimal base, which was recognized by Napier as an improvement, was probably entirely due to his suggestion. Briggs at once carried this into effect, and in 1617 brought out a table of logarithms of the first 1000 numbers to 14 places of decimals. He subsequently (in 1624) published tables of the logarithms of additional numbers and of various trigonometrical functions. His logarithms of the natural numbers are equal to those to the base 10 when multiplied by 10^8, and of the sines of angles to those to the base 10 when multiplied by 10^{12}. Other tables were brought out in 1620 by Edmund Gunther (1580–1626) another of the Gresham lecturers, who was the inventor of the words *cosine* and *cotangent*. The rapid recognition throughout Europe of the advantages of using logarithms in all practical calculations was mainly due to Briggs, and by 1630 they would seem to have come into general use. The calculation of some 20,000 logarithms which had been left out by Briggs in his tables of 1624 was performed by Vlacq and published in 1628. The *Arithmetica logarithmica* of Briggs and Vlacq are substantially the same as the existing tables: the only table founded on fresh calculations being that issued by Sang in London in 1871.

(iv) The introduction of the decimal notation was (in my opinion) due to Briggs. Stevinus had previously in 1585 used a somewhat similar notation, for he wrote a number such as 25·379 either in the form 25, 3′ 7″ 9‴ or in the form

25 ⓪ 3 ① 7 ② 9 ③;

and Napier in his essay on rods in 1617 had adopted the latter notation. These systems however only provided a concise way of stating results, and neither Stevinus nor Napier made any use of the sign as an operative form. The same notation occurs however in the tables published by Briggs in 1617, and

would seem to have been used by him in all his works, and though it is difficult to speak with absolute certainty I have myself but little doubt that he there employed the symbol as an operative form and not merely as a concise way of stating a result. At any rate in Napier's posthumous *Constructio* published in 1619 it is defined and used systematically as an operative form. Now this work was written after consultation with Briggs, circ. 1615—6, and was probably revised by Briggs before it was issued, and the only doubtful point is whether the credit of that part of the work should be given to Napier or to Briggs. Of course it is possible that Napier discovered it in the last year of his life; but looking at all the surrounding circumstances I think it is much more likely that its invention is due to Briggs and was communicated by him to Napier.

Napier wrote the point in the form now adopted, but Briggs underlined the decimal figures, and would have printed the above number as $25\underline{379}$. Later writers added another line and wrote it $25|\underline{379}$; nor was it till the beginning of the eighteenth century that the point came into general use and it was written as 25·379.

CHAPTER XII.

THE MATHEMATICS OF THE RENAISSANCE. 1450–1637.

SECTION 1. *The development of syncopated algebra and trigonometry.*

SECTION 2. *The development of symbolic algebra.*

SECTION 3. *The origin of the more common symbols in algebra.*

THE last chapter is a digression from the chronological arrangement to which as far as possible I have throughout adhered, but I trust by references in this chapter to keep the order of events and discoveries quite clear. I return now to the general history of mathematics in western Europe. Mathematicians had barely assimilated the knowledge obtained from the Arabs, including their translations of Greek writers, when the refugees who escaped from Constantinople after the fall of the eastern empire brought the original works and the traditions of Greek science into Italy. Thus by the middle of the fifteenth century the chief results of Greek and Arabian mathematics were accessible to European students. The invention of printing about that time rendered their dissemination comparatively easy. It is almost a truism to remark that until printing was introduced a writer appealed to a very limited class of readers, but we are perhaps apt to forget that when a Greek or mediaeval writer "published" a work the results were known to only a few of his contemporaries.

The introduction of printing marks the beginning of the modern world in science as in politics; for it was contemporaneous with the assimilation by the indigenous European

school (which was born from scholasticism and the history of which was traced in chapter VIII.) of the results of the Indian and Arabian schools (whose history and influence were traced in chapters IX. and X.) and of the Greek schools (the history of which was traced in chapters II. to V.).

The last two centuries of this period of our history which may be described as the renaissance were distinguished by great mental activity in all branches of learning. The creation of a fresh group of universities (including those in Scotland) of a somewhat less complex type than the mediaeval universities above described testify to the general desire for knowledge. The discovery of America in 1492 and the discussions that preceded the Reformation flooded Europe with new ideas which by the invention of printing were widely disseminated, but the advance in mathematics was perhaps even more marked than that in literature and politics.

During the first part of this time the attention of mathematicians was almost wholly directed to syncopated algebra and trigonometry: the treatment of these subjects is discussed in the first section of this chapter. The middle years of the renaissance were distinguished by the development of symbolic algebra: this is treated in the second section of this chapter. The close of the sixteenth century saw the creation of the science of dynamics: this forms the subject of the first section of chapter XIII. About the same time and in the early years of the seventeenth century considerable attention was paid to pure geometry: this forms the subject of the second section of chapter XIII.

The development of syncopated algebra and trigonometry.

Amongst the many distinguished writers of this time *Johann* Regiomontanus* was the earliest and one of the most

* For an account of his writings, see *Regiomontanus*, ein geistiger Vorläufer des Copernicus, by Ziegler, Dresden, 1874; and for an account of his life, see the memoir by Gassendi, The Hague, 1654.

able. He was born at Königsberg on June 6, 1436, and died at Rome on July 6, 1476. His real name was Johann Müller, but following the custom of that time he issued his publications under a Latin pseudonym. To his friends, his neighbours, and his tradespeople he may have been Johann Müller, but the literary and scientific world knew him as Regiomontanus just as they knew Zepernik as Copernicus, and Schwarzerd as Melanchthon. It seems to me as pedantic as it is confusing to refer to an author by his actual name when he is universally recognized under another: I shall therefore in all cases as far as possible use that title only, whether latinized or not, by which a writer is generally known.

Regiomontanus studied mathematics at the university of Vienna, then one of the chief centres of mathematical studies in Europe, under Purbach* who was professor there. His first work, done in conjunction with Purbach, consisted of an analysis of the *Almagest*. In this the trigonometrical functions *sine* and *cosine* were used and a table of natural sines was introduced. Purbach died before the book was finished: it was finally published at Venice, but not till 1496. As soon as this was completed Regiomontanus wrote a work on astrology, which contains some astronomical tables and a table of natural tangents: this was published in 1490.

Leaving Vienna in 1462, Regiomontanus travelled for some time in Italy and Germany; and at last in 1471 settled for a few years at Nuremberg, where he established an observatory, opened a printing press, and probably lectured. Thence he moved to Rome on an invitation from Sixtus IV. who wished him to reform the calendar. He was assassinated shortly after his arrival, at the age of 40.

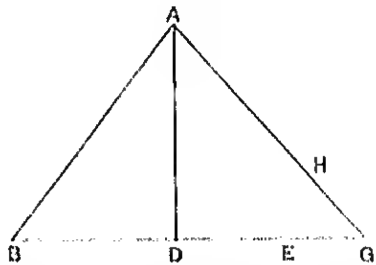
Regiomontanus was among the first to take advantage of

* *Georg Purbach* was born near Linz on May 30, 1423 and died at Vienna on April 8, 1461. He wrote a work on planetary motions which was published in 1460; an arithmetic, published in 1511; a table of eclipses, published in 1514; and a table of natural sines, published in 1541.

the recovery of the original texts of the Greek mathematical works to make himself acquainted with the methods of reasoning and results there used; the earliest notice in modern Europe of the algebra of Diophantus is a remark of his that he had seen a copy of it at the Vatican. He was also well read in the works of the Arab mathematicians.

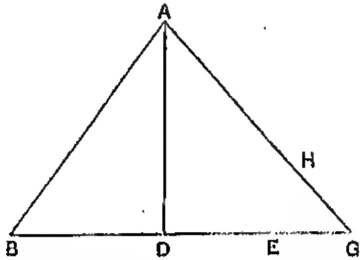
The fruit of this study was shown in his *De triangulis* written in 1464. This is a complete and systematic exposition of trigonometry, plane and spherical, though the only trigonometrical functions introduced are those of the sine and cosine. It is divided into five books. The first four are given up to plane trigonometry, and in particular to determining triangles from three given conditions. The fifth book is devoted to spherical trigonometry. The work was printed in 5 volumes at Nuremberg in 1533, nearly a century after the death of Regiomontanus.

As an example of the mathematics of this time I quote one of his propositions at length. It is required to determine a triangle when the difference of two sides, the perpendicular on the base, and the difference between the segments into which the base is thus divided are given (book ii., prop. 23). The following is the solution given by Regiomontanus. "Sit talis



triangulus ABG, cujus duo latera AB et AG differentia habeant nota HG, ductaque perpendiculari AD duorum casuum BD et DG, differentia sit EG: hae duae differentiae sint datae, et ipsa

perpendicularis AD data. Dico quod omnia latera trianguli nota concludentur. Per artem rei et census hoc problema absolvemus. Detur ergo differentia laterum ut 3, differentia casuum 12, et perpendicularis 10. Pono pro basi unam rem, et pro aggregato laterum 4 res, nae proportio basis ad congeriem



laterum est ut HG ad GE, scilicet unius ad 4. Erit ergo BD $\frac{1}{2}$ rei minus 6, sed AB erit 2 res demptis $\frac{3}{2}$. Duco AB in se producuntur 4 census et $2\frac{1}{4}$ demptis 6 rebus. Item BD in se facit $\frac{1}{4}$ census et 36 minus 6 rebus: huic addo quadratum de 10 qui est 100. Colliguntur $\frac{1}{4}$ census et 136 minus 6 rebus æquales videlicet 4 censibus et $2\frac{1}{4}$ demptis 6 rebus. Restaurando itaque defectus et auferendo utrobique æqualia, quemadmodum ars ipsa præcipit, habemus census aliquot æquales numero, unde cognitio rei patebit, et inde tria latera trianguli more suo innotescet."

To explain the language of the proof I should add that

sum of the sides will be $4x$. Therefore BD will be equal to $\frac{1}{2}x - 6$ ($\frac{1}{2}$ rei minus 6), and AB will be equal to $2x - \frac{3}{2}$ (2 res demptis $\frac{3}{2}$); hence AB^2 (AB in se) will be $4x^2 + 2\frac{1}{4} - 6x$ (4 census et $2\frac{1}{4}$ demptis 6 rebus), and BD^2 will be $\frac{1}{4}x^2 + 36 - 6x$. To BD^2 he adds AD^2 (quadratum de 10) which is 100, and states that the sum of the two is equal to AB^2. This he says will give the value of x^2 (census), whence a knowledge of x (cognitio rei) can be obtained, and the triangle determined.

To express this in the language of modern algebra we have

$$AG^2 - DG^2 = AB^2 - DB^2,$$

$$\therefore \quad AG^2 - AB^2 = DG^2 - DB^2,$$

but by the given numerical conditions

$$AG - AB = \tfrac{1}{4}(DG - DB),$$

$$\therefore \quad AG + AB = 4\,(DG + DB) = 4x.$$

Therefore $\qquad AB = 2x - \frac{3}{2}$, and $BD = \frac{1}{2}x - 6$.

Hence $\qquad (2x - \frac{3}{2})^2 = (\frac{1}{2}x - 6)^2 + 100.$

From which x can be found, and all the elements of the triangle determined.

It is worth noticing that Regiomontanus merely aimed at giving a general method, and the numbers are not chosen with any special reference to the particular problem. Thus he does not attempt in the figure given to make GE anything like four times as long as GH, and since x is ultimately found to be equal to $\sqrt{37}$, the point D really falls outside the base. The letters ABC used to denote the triangle are of course derived from the Greek order of the letters.

Some of the solutions which he gives are unnecessarily complicated, but it must however be remembered that algebra and trigonometry were still only in the rhetorical stage of development, and when every step of the argument is expressed in words at full length it is by no means easy to realize all that is contained in a formula.

It will be observed from the above example that Regiomontanus did not hesitate to apply algebra to the solution of geometrical problems. Another illustration of this is in his

solution of a problem which appears in Brahmagupta's *Siddhanta*. The problem was to construct a quadrilateral, having its sides of given lengths, which should be inscribable in a circle. The solution, which he effected by means of trigonometry, was published by Murr at Nuremberg in 1786.

The *De triangulis* which is the earliest modern systematic trigonometry was immediately followed by an algebra and arithmetic entitled *Algorithmus demonstratus*, published at Nuremberg in 1534. This book contains the earliest known instances of the use of letters to denote known as well as unknown quantities, and they are used in the demonstrations of the rules of arithmetic as well as of algebra. It is probable that the book was not generally known until it was printed in 1534.

I may however note here that it constantly happens in the history of mathematics that improvements in notation or discoveries are made long before they are generally adopted or their advantages realized. Thus the same thing may be discovered over and over again, and it is not until the general standard of knowledge requires some such improvement, or it is enforced by some one whose zeal or attainments compel attention, that it is adopted and becomes part of the science. We shall see that Regiomontanus in using letters or symbols to represent any quantities which occur in analysis was more than a century in advance of his contemporaries. A similar notation was tentatively introduced by other and later mathematicians, and it was not until it had been thus independently discovered several times that it came into general use. In point of time the book is the earliest syncopated algebra written in Europe.

Besides these works Regiomontanus left behind three tracts on astronomy, and a large number of letters which afford much valuable information on the mathematics of that age. The latter were collected and edited by Murr, Nuremberg, 1786.

Although all the algebraists of this time were Italians, yet algorism—the art of practical arithmetic—was also studied

in England and Germany. The German algorists were less fettered by precedent and tradition, and introduced some improvements in notation which were hardly likely to occur to an Italian. Of these the most prominent were the introduction of the current symbols for addition, subtraction, and equality.

The earliest instance of the use of the signs + and − of which we have any knowledge occur in the *Mercantile arithmetic** of *Johann* Widman of Eger, published at Leipzig in 1489. They are however not used by him as symbols of operation, but apparently merely as marks signifying excess or deficiency. The corresponding use of the word surplus or overplus (see Levit. xxv. 27, and 1 Maccab. x. 41) is still retained in commerce. It is noticeable that with very few exceptions these signs only occur in practical mercantile questions: hence it has been conjectured that they were originally warehouse marks. Some kinds of goods were sold in a sort of wooden chest called a *lagel*, which when full was apparently expected to weigh roughly either three or four *centners*; if one of these cases was a little lighter, say 5 lbs., than four centners Widman describes it as 4c − 5 lbs.: if it was 5 lbs. heavier than the normal weight it is described as 4c + 5 lbs.: and there are some slight reasons for thinking that these marks were chalked on to the chests as they came into the warehouse. The symbols are used as if they would be familiar to his readers. It will be observed that the vertical line in the symbol for excess printed above is somewhat shorter than the horizontal line. This is also the case with Stifel and most of the early writers who used the symbol: some presses continued to print it in this, its earliest form, up to the end of the seventeenth century. Xylander on the other hand in 1575 has the vertical bar much longer than the horizontal line, and the symbol is something like †. We

* See an article by Prof. de Morgan in the *Cambridge Philosophical Transactions*, 1864, p. 203; and another by Prince Boncompagni in the *Bullettino di bibliografia*... for 1876, p. 188. Widman was probably a physician.

infer that the more usual case was for a chest to weigh a little less than its reputed weight, and as the sign – placed between two numbers was a common symbol to signify some connection between them, that seems to have been taken as the standard case, while the vertical bar was originally a small mark superadded on the sign – to distinguish the two symbols.

I am far from saying that this account of the origin of our symbols for *plus* and *minus* is established beyond doubt, but it is the most plausible that has yet been advanced. Another suggested derivation is that + is a contraction of 𝔓 the initial letter in Old German of plus, while – is the limiting form of *m* (for minus) when written rapidly. Prof. de Morgan in his *Arithmetical Books*, London, 1847, p. 19, proposed another derivation. The Hindoos sometimes used a dot to indicate subtraction, and this dot might be thought have been elongated into a bar, and thus give the sign for *minus*; while the origin of the sign for *plus* was derived from it by a superadded bar as explained above; but I take it that at a later time he abandoned this theory for what has been called the warehouse explanation.

I should perhaps here add that till the close of the sixteenth century the sign + connecting two quantities like a and b was also used in the sense that if a were taken as the answer to some question one of the given conditions would be too little by b. This was a relation which constantly occurred in solutions of questions by the rule of false assumption (see e.g. p. 95).

Lastly I would repeat again that these signs in Widman are only abbreviations and not symbols of operation; he attached little or no importance to them, and would no doubt have been amazed if he had been told that their introduction was preparing the way for a complete revolution of the processes used in algebra.

Regiomontanus did a great deal to develope astronomy and trigonometry, but his algebra was not published till 1534; Widman's work was not known outside Germany; and it is to Pacioli that we owe the introduction into Italy of syncopated

algebra; that is the use of abbreviations for certain of the more common algebraical quantities and operations, but where in using them the rules of syntax are observed.

Lucas Pacioli, sometimes known as *Lucas di Burgo*, and sometimes, but more rarely, as *Lucas Paciolus* was born at Burgo in Tuscany about the middle of the fifteenth century. We know very little of his life except that he was a minorite friar, and that he lectured on mathematics at Rome, Pisa, Venice, and Milan; at the latter city he was the first occupant of a chair of mathematics founded by Sforza: he died at Florence about the year 1510.

His chief work was printed at Venice in 1494 and is termed *Summa de arithmetica, geometria, proporzioni e proporzionalita*. It consists of two parts, the first dealing with arithmetic and algebra, the second with geometry. This was the earliest printed book on arithmetic and algebra and it is founded on the writings of Leonardo of Pisa.

In the arithmetic he gives rules for the four simple processes, and a method for extracting square roots. He deals pretty fully with all questions connected with mercantile arithmetic, in which he works out numerous examples, and in particular discusses at great length bills of exchange and the theory of book-keeping by double entry. This part was the first systematic exposition of algoristic arithmetic and has been already alluded to in chapter XI. It and the similar work by Tartaglia are the two standard authorities on the subject. Most of the problems are solved by the method of false assumption (see p. 95), but there are numerous numerical mistakes.

The following example will serve as an illustration of the kind of arithmetical problems discussed. "I buy" says he "for 1440 ducats at Venice 2400 sugar loaves, whose nett weight is 7200 lire; I pay as a fee to the agent 2 per cent.; to the weighers and porters on the whole, 2 ducats; I afterwards spend in boxes, cords, canvas, and in fees to the ordinary packers in the whole, 8 ducats; for the tax or octroi duty on the first amount, 1 ducat per cent.; afterwards for duty and

tax at the office of exports, 3 ducats per cent.; for writing directions on the boxes and booking their passage, 1 ducat; for the bark to Rimini, 13 ducats; in compliments to the captains and in drink for the crews of armed barks on several occasions, 2 ducats; in expenses for provisions for myself and servant for one month, 6 ducats; for expenses for several short journeys over land here and there, for barbers, for washing of linen and of boots for myself and servant, 1 ducat; upon my arrival at Rimini I pay to the captain of the port for port dues in the money of that city, 3 lire; for porters, disembarkation on land, and carriage to the magazine, 5 lire; as a tax upon entrance, 4 soldi a load which are in number 32 (such being the custom); for a booth at the fair, 4 soldi per load; I further find that the measures used at the fair are different to those used at Venice, and that 140 lire of weight are there equivalent to 100 at Venice, and that 4 lire of their silver coinage are equal to a ducat of gold. I ask therefore, at how much I must sell a hundred lire Rimini, in order that I may gain 10 per cent. upon my whole adventure, and what is the sum which I must receive in Venetian money?"

In the algebra he finds expressions for the sum of the squares and cubes of the first n natural numbers. The larger half of this part of the book is taken up with simple and quadratic equations, and problems on numbers which lead to such equations. It may be noticed that all his equations are numerical, i.e. he did not rise to the conception of representing known quantities by letters as is the case in modern algebra: but M. Libri gives two instances in which in a proportion he represents a number by a letter. Pacioli confines his attention to the positive roots of equations. He follows the Arabs in calling the unknown quantity the *thing*, in Italian *cosa**, or in Latin *res*, and sometimes denotes it by *R* or *Rj*. He calls the square of it *zensus* or *census* and sometimes denotes it by *Z*; similarly the cube of it or *cuba* is sometimes represented by *C*; but no abbreviations are used for the

* Hence algebra was sometimes known as the cossic art.

higher powers, thus the fourth power or *censo di censo* is written at length and so on. He indicates addition and equality by the initial letters of the words *plus* and *aequalis*, but he generally evades the introduction of a symbol for *minus* by writing his quantities on that side of the equation which makes them positive, though in one or two places he denotes it by *de* for demptus. This marks the commencement of syncopated algebra, for the work of Regiomontanus was not printed till 1534.

The following is the rule given by Pacioli for solving a quadratic equation of the form $x^2 + x = a$; it is rhetorical and not syncopated, and will thus also serve to illustrate the inconvenience of that method.

> "Si res et census numero coaequantur, a rebus
> dimidio sumpto censum producere debes,
> addereque numero, cujus a radice totius
> tolle semis rerum, census latusque redibit."

There is nothing striking in the results he arrives at in the second or geometrical part of the work; nor in two other tracts on geometry which he wrote and which were printed at Venice in 1508 and 1509. It may however be noticed that like Regiomontanus he applied algebra to aid him in investigating the geometrical properties of figures.

The following problem will illustrate the kind of geometrical question he attacked. The radius of the inscribed circle of a triangle is 4 inches, and the segments into which one side is divided by the point of contact are 6 inches and 8 inches respectively. Determine the other sides. To solve this it is sufficient to remark that $rs = \Delta = \sqrt{s(s-a)(s-b)(s-c)}$ which gives $4s = \sqrt{s \times (s-14) \times 6 \times 8}$, hence $s = 21$; $\therefore$ the required sides are $21-6$ and $21-8$, i.e. 15 and 13. But Pacioli makes no use of these formulae (with which he was acquainted) but gives an elaborate geometrical construction and then uses algebra to find the lengths of various segments of the lines he wants. The work covers some pages of his book and is too

long for me to reproduce here, but the following analysis of it will afford sufficient materials for its reproduction. Let ABC be the triangle, D, E, F the points of contact of the sides, and O the centre of the given circle. Let H be the point of intersection of OB and DF, and K that of OC and DE. Let L and M be the feet of the perpendiculars drawn from E and F on BC. Draw EP parallel to AB and cutting BC in P. Then Pacioli determines in succession the magnitudes of the following lines: (i) OB, (ii) OC, (iii) FD, (iv) FH, (v) ED, (vi) EK. He then forms a quadratic equation from the solution of which he obtains the values of MB and MD. Similarly he finds the values of LC and LD. He now finds in succession the values of EL, FM, EP, and LP; and then by similar triangles obtains the value of AB which is 13. This proof was, even sixty years later, quoted by Cardan as "incomparably simple and excellent, and the very crown of mathematics." I have mentioned it chiefly to illustrate how involved and inelegant were the methods of even the greatest mathematicians of this time. The problems enunciated are very similar to those in the *De triangulis* of Regiomontanus.

An account of *Nicolaus* **Copernicus**, born at Thorn on Feb. 19, 1473 and died at Frauenberg on May 7, 1543, and his conjecture that the earth and planets all revolved round the sun belong to astronomy rather than to mathematics. I may however add that Copernicus wrote a short text-book on trigonometry published at Wittenberg in 1542 which is very clear though it contains nothing new. It is evident from this and his astronomy that he was well read in the literature of mathematics, and was himself a mathematician of considerable power. I describe his statement as to the motion of the earth as a conjecture because he advocated it only on the ground that it gave a simple explanation of natural phenomena. Galileo in 1632 was the first to try to supply anything like a proof of this hypothesis.

The sign now used to denote equality was introduced by *Robert* **Recorde**. Recorde was born at Tenby in Pembroke-

shire about 1510 and died at London in 1558. He entered at Oxford, and obtained a fellowship at All Souls' College in 1531; he then migrated to Cambridge, where he took a degree in medicine in 1545. He then returned to Oxford and lectured there, but finally settled in London and became physician to Edward VI. and Mary. I have already alluded (see pp. 169 and 174) to his arithmetic published in 1540. A few years later, in 1557, he wrote an algebra in which he showed how the square root of algebraical expressions could be extracted; he also wrote an astronomy. These works give a very clear view of the knowledge of the time.

The sign = for equality is first used in his arithmetic, and he says he selected that particular symbol because than two parallel straight lines no two things can be more equal. M. Charles Henry has however pointed out in the *Revue Archéologique* for 1879 that it is a not uncommon abbreviation for the word *est* in medieval manuscripts; and this would seem to point to a much more probable origin.

Recorde also employed the signs + for *plus* and − for *minus*; and there are faint traces of his having used them as symbols of operation and not as mere abbreviations.

Michael Stifel sometimes known by the Latin name of *Stifelius* was born at Esslingen in 1486 and died at Jena on April 19, 1567. He was originally an Augustine monk, but he accepted the doctrines of Luther, of whom he was a personal friend. He tells us in his algebra that his conversion was finally determined by noticing that the pope Leo X. was the beast mentioned in the Revelation. To show this it was only necessary to add up the numbers represented by the letters in Leo decimus (the *m* had to be rejected since it clearly stood for *mysterium*) and the result amounts to exactly ten less than 666, thus distinctly implying that it was Leo the tenth. Luther accepted his conversion, but frankly told him he had better clear his mind of any nonsense about the number of the beast.

Unluckily for himself Stifel did not act on this advice. Be-

lieving that he had discovered the true way of interpreting the biblical prophecies, he announced that the world would come to an end on Oct. 3rd, 1533. The peasants of Holzdorf, of which place he was pastor, knowing of his scientific reputation accepted his assurance on this point. Some gave themselves up to religious exercises, others wasted their goods in dissipation, but all abandoned their work. When the day foretold had passed, many of the peasants found themselves ruined: furious at having been deceived, they seized the unfortunate prophet, and he was lucky in finding a refuge in the prison at Wittenberg, from which he was after some time released by the personal intercession of Luther.

Stifel wrote a small treatise on algebra, but his chief mathematical work is his *Arithmetica integra* published at Nuremberg in 1544, with a preface by Melanchthon.

The first two books of the *Arithmetica integra* deal with surds and incommensurables, and are Euclidean in form. The third book is on algebra, and is sometimes said to have introduced the study of algebra into Germany. This however is a mistake, for it is stated in it that a large part is taken from two previous writers A. Riese and C. Rudolph, but how much of it is due to them and how much to Stifel is not known.

This work is chiefly noticeable for having called general attention to the German practice of using the signs + and − to denote addition and subtraction (see p. 174 and p. 185). There are faint traces of their being occasionally employed by Stifel as symbols of operation and not only as abbreviations; whether this application of them was new or whether it was taken from Rudolph is doubtful, but the prevalent opinion is that it was original. Stifel also introduced the sign of $\surd$ for the square root, the symbol being a corruption of the initial letter of the word *radix*. Like Pacioli he used abbreviations for the Italian words which represent the unknown quantity and its powers. It would seem however that he made a further step forward, and that in at least one case when there were several unknown quantities he represented them respectively by the letters A,

B, *C*, &c. It used to be said that he was the real inventor of logarithms, but it is now certain that this opinion was due to a misapprehension of a passage in which he compares geometrical and arithmetical progressions.

Niccola Fontana, generally known as *Nicholas* Tartaglia that is Nicholas the stammerer, was born at Brescia in 1500 and died at Venice on December 14, 1559. After the capture of the town by the French in 1512 most of the inhabitants took refuge in the cathedral, and were there massacred by the soldiers. His father who was a postal messenger at Brescia was amongst the killed. The boy himself had his skull split through in three places, while both his jaws and his palate were cut open. He was left for dead, but his mother got into the cathedral, and finding him still alive managed to carry him off. Deprived of all resources she recollected that dogs when wounded always licked the injured place, and to that remedy he attributed his ultimate recovery; the injury to his palate produced an impediment in his speech from which he received his nickname. His mother managed to teach him how to read and write, but so poor were they that he tells us they could not afford to buy paper, and he was obliged to make use of the tombstones on which to work his exercises.

He commenced his public life by lecturing at Verona, but he was appointed at some time before 1535 to a chair of mathematics at Venice, where he was living when he became famous through his acceptance of a challenge from a certain *Antonio del Fiori*. According to this challenge Fiori and Tartaglia were to deposit certain stakes with a notary, and whoever could solve the most problems out of a collection of thirty propounded by the other was to get the stakes, thirty days being allowed for the solution of the questions proposed.

Fiori had learnt from his master, one *Scipione Ferro* (who died at Bologna in 1525), an empirical solution of a cubic equation of a certain form. This solution was previously unknown in Europe, and it is probable that Ferro had found the result in an Arab work. Tartaglia knew that his adversary was thus

prepared, and suspecting that the questions proposed to him would all depend on the solution of cubic equations set himself the problem to find a general solution. His solution is believed to have depended on a geometrical construction (see p. 200), but led to the formula which is often, but unjustly, described as Cardan's.

When the contest took place the questions proposed to Tartaglia were as he had suspected all reducible to the solution of a cubic equation, and he succeeded within two hours in bringing them to particular cases of the equation $x^3+px=q$, of which he knew the solution. His opponent failed to solve any of the problems proposed to him. Tartaglia was therefore the conqueror; he subsequently composed some verses commemorative of his victory.

His chief works are as follows. (i) His *Nova scienza*, published in 1537, in which he investigated the fall of bodies under gravity and the range of a projectile: he stated that the latter was a maximum when the angle of projection was 45°, but this seems to have been a lucky guess. (ii) An arithmetic published in two parts in 1556, which is verbose but able. (iii) A treatise on numbers, published in four parts in 1560, and sometimes treated as a continuation of the arithmetic: in this he showed how the coefficients of x in the expansion of $(1+x)^n$, n being a positive integer, could be calculated from those in the expansion of $(1+x)^{n-1}$ for the cases when n is equal to 2, 3, 4, 5, or 6. It is believed that he also wrote a treatise on algebra and the solution of cubic equations, but no copy is now extant. The other works were collected into a single edition and re-published at Venice in 1606.

This treatise on arithmetic and numbers is the chief authority for our knowledge of the early Italian algorism. It is verbose, but gives a clear account of all the different arithmetical methods then in use, and has numerous historical notes which as far as we can judge are reliable, and are the authorities for many of the statements in the last chapter. Like Pacioli he gives an immense number of questions on every kind of problem which could occur in mercantile arithmetic,

and makes several attempts to frame algebraical formulæ suitable for particular problems.

These problems give incidentally a good deal of information as to the ordinary life and commercial customs of the time. Thus we find that the interest demanded on first class security in Venice ranged from 5 to 12 per cent. a year; while the interest on commercial transactions ranged from 20 per cent. a year upwards. Tartaglia illustrates the evil effects of the law forbidding usury by the manner in which it was evaded in farming. Farmers who were in debt were forced by their creditors to sell all their crops immediately after the harvest, the market being thus glutted the price obtained was very low, and the money lenders purchased the corn in open market at an extremely cheap rate. The farmers then had to borrow their seed-corn on condition that they replaced an equal quantity, or paid the then price of it, in the month of May, i.e. when corn was dearest. Again Tartaglia, who had been asked by the magistrates at Verona to frame for them a sliding scale by which the price of bread would be fixed by that of corn, enters into a discussion on the principles which it was then supposed should regulate it. In another place he gives the rules at that time current for preparing medicines.

Pacioli had given in his arithmetic some questions of an amusing character, and Tartaglia imitated him by inserting a large collection of mathematical puzzles. He half apologizes for introducing them by saying that it was not uncommon at dessert to propose arithmetical questions to the company by way of amusement, and he therefore adds some suitable problems. Questions on how to guess a number thought of by one of the company, or the relationships caused by the marriage of relatives, or difficulties arising from inconsistent bequests may perhaps pass muster as amusing; but it certainly seems a curious way of entertaining a pretty woman to insist on an answer to so absurd a question as what would 10 be if 4 were 6, a problem on which he evidently prided himself.

He gives several questions such as the following. "There

are three men, young, handsome, and gallant, who have three beautiful ladies for wives: all are jealous, as well the husbands of the wives as the wives of the husbands.... They find on the bank of a river, over which they have to pass, a small boat which can hold no more than two persons. How can they pass so as to give rise to no jealousy?"

Other problems are like the following. "A ship on board of which there are fifteen Turks and fifteen Christians, encounters a storm and the pilot declares, that in order to save the ship one-half of the passengers must be thrown into the sea: the men are placed in a circle, and it is agreed that every ninth man must be cast overboard, reckoning from a certain point. In what manner must the men be arranged, so that the lot may fall exclusively upon the Turks?"

The following is a sample of another class of puzzles. "Three persons have robbed a gentleman of a vessel of balsam, containing 24 ounces; and whilst running away they meet in a wood with a glass-seller of whom in a great hurry they purchase three vessels. At last on reaching a place of safety they wish to divide the booty, but they find that their vessels contain 5, 11 and 13 ounces respectively. How can they divide the spoil into equal portions?" Problems like this can only be worked out by trial: there are several solutions, of which one is as follows:

	oz.	oz.	oz.	oz.
The vessels can contain	24 ...	13 ...	11 ...	5
Their contents originally are	24 ...	0 ...	0 ...	0
First make their contents ...	0 ...	8 ...	11 ...	5
Second " " ...	16 ...	8 ...	0 ...	0
Third " " ...	16 ...	0 ...	8 ...	0
Fourth " " ...	3 ...	13 ...	8 ...	0
Fifth " " ...	3 ...	8 ...	8 ...	5
Last " " ...	8 ...	8 ...	8 ...	0

These problems form the basis of the collections of mathematical recreations by Bachet de Méziriac*, 1624; Ozanam,

* *Claude Gaspard Bachet de Méziriac*, born at Bourg in 1581 and

1694; and Montucla, 1754. The latter was translated with additions by Hutton, and the second edition in 4 vols. was issued in London in 1814.

The life of Tartaglia was embittered by a quarrel with his contemporary Cardan who having under a pledge of secrecy obtained Tartaglia's solution of a cubic equation, published it.

Hieronymus (or *Girolamo*) **Cardan*** was born at Pavia on Sept. 24, 1501 and died at Rome on Sept. 21, 1576. His career is an account of the most extraordinary and inconsistent acts. A gambler, if not a murderer, he was also the ardent student of science, solving problems which had long baffled all investigation; at one time his life was devoted to intrigues which were a scandal even in the sixteenth century, at another he did nothing but rave on astrology, and yet at another he declared that philosophy was the only subject worthy of man's attention. His was the genius that was closely allied to madness.

He was the illegitimate son of a lawyer of Milan, and was educated at the universities of Pavia and Padua. After taking his degree he commenced life as a doctor, and practised his profession at Sacco and Milan from 1524 to 1550; it was during this period that he studied mathematics and published his chief works. After spending some years in travelling he returned to Milan as professor of mathematics, and was shortly elected to

died in 1638, wrote *Problèmes plaisants*, 1612 and 1624; *Les éléments arithmétiques*, which exists in manuscript; and a translation of Diophantus, 1621. *Jacques Ozanam*, born at Bouligneux in 1640 and died in 1717, left numerous works of which the only one worth mentioning is his *Récréations mathématiques et physiques* 2 vols. 1694. *Jean Étienne Montucla*, born at Lyons in 1725 and died in Paris in 1799, edited and revised Ozanam's mathematical recreations. His history of attempts to square the circle, 1754, and history of mathematics to the end of the seventeenth century in 2 volumes, 1758, are interesting and valuable works. The second edition of the latter in 4 volumes, 1799 (the fourth volume is by Lalande), forms the basis of most subsequent works on the subject.

* There is an admirable account of his life in the *Nouvelle Biographie générale*, by V. Sardou. Cardan left an autobiography of which an analysis was published by H. Morley, London, 1854 (2 volumes).

the chair at Bologna. Here he divided his time between debauchery, astrology, and mechanics. It is said that about 1562 he was imprisoned for heresy on account of his having published the horoscope of Christ, and when released he found himself so generally detested that he determined to resign his chair. At any rate he left Bologna in 1563, and shortly afterwards moved to Rome. His two sons were as wicked and passionate as himself: the eldest was about this time executed for poisoning his wife, and the younger having committed some offence, Cardan in a fit of rage cut off his ears; for this scandalous outrage he suffered no punishment as the pope Gregory XIII. took him under his protection. Cardan was the most distinguished astrologer of his time, and when he settled at Rome he received a pension in order to secure his services as astrologer to the papal court. This proved fatal to him, for having foretold that he should die on a particular day he felt obliged to commit suicide in order to keep up his reputation.

The chief mathematical work of Cardan is the *Ars magna* published at Nuremberg in 1545. Cardan was much interested in the contest between Tartaglia and Fiori, and as he had already begun writing this book he asked Tartaglia to communicate his method of solving a cubic equation. Tartaglia refused, whereon Cardan abused him in the most violent terms, but shortly afterwards wrote saying that a certain Italian nobleman had heard of Tartaglia's fame and was most anxious to meet him, and begged him to come to Milan at once. Tartaglia came, and though he found no nobleman awaiting him at the end of his journey, he yielded to Cardan's importunity and gave him the rule he wanted, Cardan on his side taking a solemn oath that he would never reveal it, and would not even commit it to writing in such a way that after his death any one could understand it. Cardan asserts that he was merely given the result, and obtained the proof himself, but this is doubtful. He seems to have at once taught the method, as one of his pupils Ferrari reduced the equation of

the fourth degree to a cubic and so solved it. When the *Ars magna* was published in 1545 the breach of faith was made manifest. Tartaglia was not unnaturally very angry, and after an acrimonious controversy he sent a challenge to Cardan to take part in a mathematical duel. The preliminaries were settled, and the place of meeting was to be a certain church in Milan, but when the day arrived Cardan failed to appear, and sent Ferrari in his stead. Tartaglia was victorious; but the friends of Cardan interfered, and Tartaglia was fortunate in escaping with his life. Not only was Cardan successful in his fraud, but modern writers generally attribute the solution to him, so that Tartaglia has not even that posthumous reputation which is at least his due.

The *Ars magna* is the third earliest printed book on algebra, and it is a great advance on any algebra previously published. Hitherto algebraists had confined their attention to those roots of equations which were positive. Cardan discussed negative and even imaginary roots, and proved that the latter would always occur in pairs, though he declined to commit himself to any explanation as to the meaning of these "sophistic" quantities which he said were ingenious though useless. Discussing cubic equations he showed that if the three roots were real, his solution gave them in a form which involved imaginary quantities. Except for the somewhat similar researches of Bombelli a few years later (see p. 203), the theory of imaginary quantities received little further attention from mathematicians until Euler took the matter up after the lapse of nearly two centuries. Gauss first put the subject on a systematic and scientific basis, introduced the notation of complex variables, and used the symbol i to denote the square root of -1: the modern theory is chiefly based on his researches.

Cardan found the relations connecting the roots with the coefficients of an equation. He was also aware of the principle that underlies Descartes' "rule of signs," but as he followed the then universal custom of writing his equations as the equality of two expressions in each of which all the terms were

positive he was unable to express the rule concisely. He gave a method of approximating to the root of a numerical equation, founded on the fact that if a function has opposite signs when two numbers are substituted in it the equation obtained by equating the function to zero will have a root between those two numbers.

Cardan's analysis of cubic equations seems to have been original; and it was only for the solution that he was indebted to Tartaglia. I should add that though he takes instances of cubic equations of every possible form, the equations he considers are all numerical.

The solution given of quadratic equations is geometrical and substantially the same as that given by Alkarismi (see p. 152). The solution of a cubic equation is also geometrical, and may be illustrated by the following case which he gives in chapter XI. To solve the equation $x^3 + 6x = 20$ (or any equation of the form $x^3 + px = q$), take two cubes such that the rectangle under their respective edges is 2 (or $\frac{1}{3}p$) and the difference of their volumes is 20 (or q). Then will x be equal to the difference between the edges of the cubes. To verify this he first gives a geometrical lemma to shew that if from a line AC a portion CB be cut off then the cube on AB will be less than the difference between the cubes on AC and BC by three times the right parallelopiped whose edges are respectively equal to AC, BC and AB; which is a statement of the algebraical identity $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$: and the fact that x satisfies the equation is then obvious. To obtain the lengths of the edges of the two cubes he has only to solve a quadratic equation for which the geometrical solution previously given sufficed.

Like all the mathematicians up to this time he gives separate proofs of his rule for the different forms of equations which can fall under it. Thus he proves the rule independently for equations of the form $x^3 + px = q$, $x^3 = px + q$, $x^3 + px + q = 0$, nd $x^3 + q = px$. It will be noticed that with geometrical proofs is was almost a necessity, but be did not suspect that the sulting formulæ were general.

All Cardan's printed works were collected by Sponius and published in 10 volumes, Lyons, 1663. The mathematical works form the fourth volume. It is said that there are in the Vatican numerous manuscript note-books of his which have not yet been edited.

Shortly after Cardan came a number of mathematicians who did good work in developing the subject, but who are hardly of sufficient importance to require detailed mention here. Of these the most celebrated are perhaps Ferrari and Rheticus.

Ludovico **Ferrari**, whose name I have already mentioned in connection with the solution of a biquadratic equation, was born at Bologna in 1522 and died in 1562. His parents were poor and he was taken into Cardan's service to clean knives &c., but he was allowed to attend his master's lectures, and subsequently became his most celebrated pupil. Such work as he produced is incorporated in Cardan's *Ars Magna* or Bombelli's *Algebra*, but nothing can be definitely assigned to him except the solution of a biquadratic equation. For further details see Libri, vol. III., p. 180.

Georg Joachim **Rheticus**, born at Feldkirch on Feb. 15, 1514 and died at Kaschau on Dec. 4, 1576, was professor at Wittenberg, and subsequently studied under Copernicus whose works were produced under the direction of Rheticus. Rheticus constructed some trigonometrical tables some of which were published by his pupil Otho in 1596, and which are the basis of those still in use. They were subsequently completed and extended by Vieta and Pitiscus*. Rheticus also found the values of $\sin 2\theta$ and $\sin 3\theta$ in terms of $\sin \theta$ and $\cos \theta$.

I add here the names of some other celebrated mathematicians of about the same time, though their works did not perceptibly advance the subject and are now of little value to any save antiquarians. *Franciscus* **Maurolycus**, born at Messina of Greek parents in 1494 and died in 1575, translated numerous Latin and Greek mathematical works, and discussed

* *Bartholomäus Pitiscus* was born on Aug. 24, 1561 and died at Heidelberg, where he was professor of mathematics, on July 2, 1613.

the conics regarded as sections of a cone: his works were published at Venice in 1575. *Jean* **Borrel**, born in 1492 and died at Grenoble in 1572, wrote an algebra founded on that of Stifel and a history of the quadrature of the circle: his works were published at Lyons in 1559. *Wilhelm* **Xylander**, born at Augsburg on Dec. 26, 1532 and died at Heidelberg, where since 1558 he had been professor, on Feb. 10, 1576, brought out an edition of the works of Psellus in 1556; an edition of Euclid's *Elements* in 1562; an edition of the *Arithmetic* of Diophantus in 1575; and some minor works which were collected and published in 1577. *Federigo* **Commandino**, born at Urbino in 1509 and died there on Sept. 3, 1575, published a translation of the works of Archimedes in 1558; selections from Apollonius, and Pappus in 1566; Euclid's *Elements* in 1572; and selections from Aristarchus, Ptolemy, Hero, and Pappus in 1574; all being accompanied by commentaries. And lastly *Jacques* **Peletier**, born at le Mans on July 25, 1517 and died at Paris in July 1582, wrote several text-books on algebra and geometry: most of the results of Stifel and Cardan are included in the former.

About this time also several text-books were produced which if they did not extend the boundaries of the subject systematized it. In particular I may mention those of Ramus and Bombelli.

Petrus **Ramus** was born at Cuth in Picardy in 1515, and was killed at Paris at the massacre of St Bartholomew on Aug. 24, 1572. He was educated at the university of Paris, and on taking his degree he astonished and charmed the university with the brilliant declamation he delivered on the thesis that everything Aristotle had taught was false. He lectured—for it will be remembered that in early days there were no professors—first at le Mans, and afterwards at Paris; at the latter he founded the first chair of mathematics. Besides some works on philosophy he wrote treatises on arithmetic, algebra, geometry (founded on Euclid), astronomy (founded on the works of Copernicus), and physics which were long regarded on the

continent as the standard text-books on those subjects. They are collected in an edition of his works published at Bâle in 1569. For an account of his life and writings see the monographs on him by Ch. Waddington, Paris, 1855; and by C. Desmaze, Paris, 1864.

Closely following the publication of Cardan's great work, *Rafaello* Bombelli published in 1572 an algebra which is a systematic exposition of all that was then known on the subject. In the preface he alludes to Diophantus, who in spite of the notice of Regiomontanus was still unknown in Europe, and thence traces the history of the subject. He then discusses radicals real and imaginary. He next considers the theory of equations, and shews that in the irreducible case of a cubic equation the roots are all real; and he remarks that the problem to trisect a given angle is the same as that of the solution of a cubic equation. Finally he gives a large collection of problems.

Bombelli is however best known in connection with the improvement in the notation of algebra which he introduced. The symbols then ordinarily used for the unknown quantity and its powers were letters which stood for abbreviations of the words. Those most frequently adopted were R or Rj for *radix* or *res* (x), Z or C for *zensus* or *census* (x^2), C or K for *cubus*, &c. Thus x^2+5x-4 would have been written

$$1\ Z\ \text{p.}\ 5\ R\ \text{m.}\ 4$$

where *p* stands for plus and *m* for minus. Xylander, in his edition of the *Arithmetic* of Diophantus in 1575, used other letters and would have written it thus

$$1Q + 5N - 4;$$

a similar notation was employed by Fermat as late as 1670. The advance made by Bombelli was that he introduced a symbol $\overset{1}{\smile}$ for the unknown quantity, $\overset{2}{\smile}$ for its square, $\overset{3}{\smile}$ for its cube, and so on, and therefore wrote x^2+5x-4 as

$$1\ \overset{2}{\smile}\ \text{p.}\ 5\ \overset{1}{\smile}\ m.\ 4.$$

Stevinus in 1586 employed ①, ②, ③, ... in a similar way; and suggested, though he did not use, a corresponding notation

for fractional indices (see p. 217). He would have written the above expression as

$$1 \textcircled{2} + 5 \textcircled{1} - 4 \textcircled{0}.$$

But whether the symbols were more or less convenient they were still only abbreviations for words; and were subject to all the rules of syntax. They merely afforded a sort of shorthand by which the various steps and results could be expressed concisely. The next advance was the creation of symbolic algebra, and the chief credit of that is due to Vieta.

The development of symbolic algebra.

We have now reached a point beyond which any considerable development of algebra, so long as it was strictly syncopated, could hardly proceed. It is evident that Stifel and Bombelli and other writers of the sixteenth century had introduced or were on the point of introducing some of the ideas of symbolic algebra. But so far as the credit of inventing symbolic algebra can be put down to any one man we may perhaps assign it to Vieta, while we may say that Harriot and Descartes did more than any other writers to bring it into general use. It must however be remembered that it took some time before all these innovations became generally known, and they were not familiar to mathematicians until the lapse of many years after they had been published.

One of the great improvements which was employed, even if it was not invented, by Vieta, was that he denoted the known quantities by the consonants B, C, D &c. and the unknown quantities by the vowels A, E, I &c. Thus in any problem he was able to use a number of unknown quantities: in this particular point he seems to have been forestalled by Stifel (see p. 192). The present custom of using the letters at the beginning of the alphabet a, b, c &c. to represent known quantities and those towards the end, x, y, z &c. to represent the unknown quantities was introduced by Descartes in 1637.

The other improvement was this. Till this time it had been the custom to introduce new symbols to represent the

square cube etc. of quantities which had already occurred in the equations; thus if *R* stood for res or x, *Z* or *Q* stood for zensus or x^2, and *C* or *K* for cubus or x^3 &c. So long as this was the case the chief advantage of algebra was that it afforded a concise statement of results every statement of which was reasoned out. But when Vieta used *A* to denote the unknown quantity x he employed *Aq*, *Ac*, *Aqq* &c. (abbreviations for *A quadratus*, *A cubus*, &c.) to represent x^2, x^3 ... which at once shewed the connection between the different powers. A similar improvement had been previously made by Bombelli and Stevinus. Thus Vieta would write the equation

$$3BA^2 - DA + A^3 = Z,$$

as *B 3 in A quad. − D plano in A + A cubo aequatur Z solido.* It will be observed that he makes the dimensions of his constants (*B*, *D*, and *Z*) such that the equation is homogeneous. This is characteristic of all his work. It will also be noticed that he does not use a sign for equality; and in fact he employed the sign = to represent the difference between.

These two steps were almost essential to any further progress in algebra. In both of them Vieta had been forestalled, but it was his good luck in emphasizing their importance to be the means of making them generally known at a time when opinion was ripe for such an advance.

François **Viète** or **Vieta** was born in 1540 at Fontenay near la Rochelle and died in Paris on Dec. 13, 1603. He was brought up as a lawyer and practised for some time at the Parisian bar; he then became a member of the provincial parliament in Brittany; and finally in 1580 through the influence of the duke de Rohan he was made master of requests, an office attached to the parliament at Paris. The rest of his life was spent in the public service. He was a zealous catholic and a firm believer in the right divine of kings. After 1580 he gave up most of his leisure to mathematics, though his great work *Isagoge in artem analyticam* in which he explained how algebra could be applied to the solution of geometrical problems was not published till 1591.

His mathematical reputation was already considerable, when one day the ambassador from the Low Countries remarked to Henry IV. that France did not possess any geometricians capable of solving a problem which had been propounded in 1593 by his countryman Adrian Romanus* to all the mathematicians of the world and which required the solution of an equation of the 45th degree. The king thereupon summoned Vieta and informed him of the challenge. Vieta saw that the equation was satisfied by the chord of a circle (of unit radius) which subtended an angle $2\pi/45$ at the centre, and in a few minutes he gave back to the king two solutions of the problem written in pencil. In explanation of this feat I should add that Vieta had previously discovered how to form the equation connecting $\sin n\theta$ with $\sin \theta$ and $\cos \theta$. Vieta in his turn asked Romanus to give a geometrical construction to describe a circle which should touch three given circles. This was the problem which Apollonius had treated in his *De tactionibus*, a lost book which Vieta at a later time conjecturally restored. Romanus solved the problem with the aid of the conic sections, but failed to do it by Euclidean geometry. Vieta gave a Euclidean solution which so impressed Romanus that he travelled to Tours, where the French court was then settled, to make Vieta's acquaintance—an acquaintanceship which rapidly ripened into warm friendship.

Henry was much struck with the ability shown by Vieta in this matter. The Spaniards had at that time a cipher containing nearly 600 characters which was periodically changed, and which they believed it to be impossible to decipher. A despatch having been intercepted, the king gave it to Vieta, and asked him to try to read it and find the key to the system. Vieta succeeded, and for two years the French used it, greatly to their

* *Adrian Romanus*, born at Louvain on Sept. 29, 1561 and died on May 4, 1625, was professor of mathematics and medicine at the university of Louvain. He was the first to prove the usual formula for $\sin(A+B)$. The formulae for $\cos(A \pm B)$ and $\sin(A-B)$ were given by Pitiscus in his *Trigonometry* published in 1599.

profit, in the war which was then raging. So convinced was Philip II. that the cipher could not be discovered that when he found his plans known he complained to the pope that the French were using sorcery against him, "contrary to the practice of the Christian faith."

Vieta wrote numerous works on algebra and geometry. The most important are the *In artem analyticam isagoge*, Tours, 1591; the *Supplementum geometriae* and a collection of geometrical problems, Tours, 1593; and the *De numerosa potestatum*, Paris, 1600. All of these were printed for private circulation only; but they were collected by F. van Schooten and published in one volume at Leyden in 1646.

The *In artem* introduced the use of letters for both known and unknown quantities, a notation for the powers of quantities, and enforced the advantage of working with homogeneous equations. To this an appendix called *Logistica speciosa* was added on addition and multiplication of algebraical quantities, and on the powers of a binomial up to the sixth. Vieta implies that he knew how to form the coefficients of these six expansions by means of the arithmetical triangle as Tartaglia had previously done, but Pascal was the first to give the general rule (see p. 252) for forming it for any order, which is equivalent to saying that he could write down the coefficients of x in the expansion of $(1+x)^n$ if those in the expansion of $(1+x)^{n-1}$ were known; Newton was the first to give the general expression for the coefficient of x^n (see pp. 293, 324). Another appendix known as *Zeteticorum* on the solution of equations was subsequently added to the *In artem*.

A posthumous work in two books termed *De aequationum recognitione* was published in 1615 by Alexander Anderson (born at Aberdeen in 1582 and died in 1631) and completes Vieta's works on algebra. Most of this is on the theory of equations. He here showed that the first member of an algebraical equation $\phi(x)=0$ could be resolved into linear factors, and explained how the coefficients of x could be expressed as functions of the roots. He also indicated how from a given equation another

could be obtained whose roots were equal to those of the original increased by a given quantity or multiplied by a given quantity: and he used this method to get rid of the coefficient of x in a quadratic equation and of the coefficient of x^2 in a cubic equation, and was thus enabled to give the general algebraic solution of both.

His solution of a cubic equation is as follows. First reduce the equation to the form $x^3 + 3a^2x = 2b^3$. Next let $x = a^2/y - y$, and we get $y^6 + 2b^3y^3 = a^6$ which is a quadratic in y^3. Hence y can be found, and therefore x can be determined.

His solution of a biquadratic is similar to that known as Ferrari's. He first reduces the equation to the form

$$x^4 + a^2x^2 + b^3x = c^4.$$

He then takes the terms involving x^2 and x to the right-hand side, and adds $x^2y^2 + \frac{1}{4}y^4$ to each side so that the equation becomes

$$(x^2 + \tfrac{1}{2}y^2)^2 = x^2(y^2 - a^2) - b^3x + \tfrac{1}{4}y^4 + c^4.$$

He then chooses y so that the left-hand side is a perfect square. Substituting this value of y, he can take the square root of both sides, and he thus gets two quadratic equations for x, each of which can be solved.

The *De numerosa potestatum* deals with numerical equations. In this a method for approximating to the values of positive roots is given; but it is prolix and of little use. Negative roots are uniformly rejected. This work is hardly worthy of Vieta's reputation.

Vieta's trigonometrical researches are included in various tracts which are collected in Schooten's edition. Besides some trigonometrical tables he gave the general expression for the sine (or chord) of an angle in terms of the sine and cosine of its submultiples. Delambre considers this as the completion of the Arab system of trigonometry. We may take it then that from this time the results of elementary trigonometry were familiar to mathematicians.

Among Vieta's miscellaneous tracts will be found a proof that each of the famous geometrical problems of the trisection

of an angle and the duplication of the cube depend on the solution of a cubic equation. There are also several papers connected with a long and angry controversy with Clavius, in 1594, on the subject of the reformed calendar, in which he was rather severely handled.

Vieta's works on geometry are good but they contain nothing which requires mention here. He applied algebra and trigonometry to help him in investigating the properties of figures. He also, as I have already said, laid great stress on the desirability of always working with homogeneous equations, so that if a square or a cube were given it should be denoted by expressions like a^2 or b^3 and not by terms like m or n which do not indicate the dimensions of the quantities they represent. He had a lively dispute with Scaliger, on the latter publishing a solution of the quadrature of the circle, and succeeded in showing the mistake into which his rival had fallen. He gave a solution of his own which as far as it goes is correct, and stated (Schooten's edition p. 400) that the area of a square is to that of the circumscribing circle as

$$\sqrt{\tfrac{1}{2}} \times \sqrt{(\tfrac{1}{2} + \sqrt{\tfrac{1}{2}})} \times \sqrt{\{\tfrac{1}{2} + \sqrt{(\tfrac{1}{2} + \sqrt{\tfrac{1}{2}})}\}} \ldots \textit{ad inf.} : 1.$$

This is one of the earliest attempts to find the value of π by means of an infinite series. He was well acquainted with the extant writings of the Greek geometricians, and introduced the curious custom, which during the seventeenth and eighteenth centuries became fashionable, of restoring lost classical works. He himself produced a conjectural restoration of the *De tactionibus* of Apollonius.

The invention of logarithms by *John* Napier of Merchistoun in 1615, and their introduction into England by *Henry* Briggs and others, has been already mentioned in chapter XI. I only add here that Napier's attention may have been partly directed to the desirability of facilitating computations by the stupendous arithmetical efforts of some of his contemporaries, who seem to have taken a keen pleasure in surpassing one another in the extent to which they carried multiplications and divi-

sions. The trigonometrical tables which were published by Rheticus in 1596 and 1613 were calculated in a most laborious way: Vieta himself delighted in arithmetical calculations which must have taken hours or days of hard work and of which the results often served no useful purpose: while Cataldi* published in 1612 a work on numerical approximations to the value of π which involved a great deal of multiplication and division. As an illustration of these useless exercises in multiplication I may note that Caspar Schott (born in 1608 and died in 1666) having *a priori* grounds for knowing that the degrees of grace of the Virgin Mary were in number the 256th power of 2, multiplied it out in several ways; and I am glad to say for the credit of the old Jesuit that whatever be the truth of his theory his calculation is quite correct. A few years later it was shown that the same result might be obtained by writing down every way in which the words of the hexameter,

Tot tibi sunt dotes, Virgo, quot sidera caelo

could be arranged; and thus the number of fixed stars could also be determined.

In regard to Napier's other work I may again mention (see p. 172) that he invented some rods which provide a mechanical way of multiplying numbers: and I should add that in spherical trigonometry he discovered certain formulae known as Napier's analogies, and also enunciated a "rule of circular parts" for the solution of right-angled spherical triangles.

Thomas Harriot, who was born at Oxford in 1560 and died in London on July 2, 1621, did a great deal to extend and codify the theory of equations. The early part of his life was spent in adventures in America with Sir Walter Raleigh: while there he made the earliest survey of Virginia and North

* *Pietro Antonio Cataldi*, born in 1548 and died in 1626, was successively professor of mathematics at Florence, Perugia, and Bologna. He wrote several works, but is chiefly known for his invention in 1613 of the form of continued fractions, though he failed to establish any of their properties.

Carolina, the maps of these he subsequently presented to Queen Elizabeth. On his return to England he settled in London and gave up most of his time to mathematical studies. The majority of the propositions I have assigned to Vieta are to be found in his writings, but it is not certain whether they were discovered by Harriot independently of Vieta or not. In any case it is probable that Vieta had not fully realized all that was contained in the propositions he had enunciated. The full consequences of these with numerous extensions and a systematic exposition of the theory of equations was given by Harriot in his *Artis analyticae praxis*, which was first printed in 1631. The influence of the work was very great, but I do not know that discoveries of any special importance beyond those given in Vieta's works can be traced back to Harriot, and I am inclined to think that the work was founded on that of Vieta. It is however far more analytical than any algebra that preceded it, and marks a great advance both in symbolism and notation. Harriot was I believe the earliest writer who realized the advantage to be obtained by taking all the terms of an equation to one side of it. He was the first to use the signs $>$ and $<$ to represent greater than and less than. When he denoted the unknown quantity by a he represented a^2 by aa, a^3 by aaa, and so on. This is a distinct improvement on Vieta's notation. The same symbolism was used by Wallis as late as 1685, but concurrently with the modern index notation which was introduced by Descartes.

Among those who contributed most powerfully to the general adoption throughout Europe of these various improvements and additions to algorism and algebra was *William* Oughtred who was born at Eton on March 5, 1574 and died at his vicarage of Albury in Surrey on June 30, 1660. Oughtred was educated at Eton and King's College, Cambridge, of the latter of which colleges he was a fellow. He invented an abbreviated rule for multiplication which is still used. He also introduced the symbol $\times$ for multiplication, and the symbol :: in proportion; previously to his time a proportion such as $a : b = c : d$ was written as

$a - b - c - d$, but he denoted it by $a \,.\, b :: c \,.\, d$. His *Clavis mathematica* published in 1631 is a good systematic text-book on arithmetic, and it contains practically all that was then known on the subject. He also wrote a *Trigonometry* published in 1657 which is one of the earliest works containing abbreviations for *sine, cosine*, &c. This was really an important advance, but the book was neglected and soon forgotten, and it was not until Euler reintroduced contractions for the trigonometrical functions that they were generally adopted. A complete edition of Oughtred's works was published at Oxford in 1677.

We may say roughly that henceforth elementary arithmetic, algebra, and trigonometry were treated in a manner which is not substantially different from that now in use; and that the subsequent improvements introduced are additions to the subject as then known, and not a re-arrangement of the subject on new foundations.

The origin of the more common symbols in algebra.

It will perhaps be convenient if I collect here in one place the scattered remarks I have made on the introduction of the various symbols for the more common operations in algebra*.

The later Greeks (see p. 98) and the Hindoos (see p. 148) indicated *addition* by mere juxtaposition. It will be observed that this is still the custom in arithmetic, where e.g. $2\frac{1}{2}$ stands for $2 + \frac{1}{2}$. The Italian algebraists, when they gave up expressing every operation in words at full length and introduced syncopated algebra, generally denoted *plus* by its initial letter P or p, a line being sometimes drawn through the letter to show that it was a symbol of operation and not a quantity: but the practice was not uniform; Pacioli for example denoted it by $\bar{e}$, and Tartaglia by ϕ. The German and English algebraists on the other hand introduced the sign + almost as soon as

* See two articles by C. Henry in the June and July numbers of the *Revue Archéologique* for 1879.

they used algorism, but they spoke of it as *signum additorum* and employed it only to denote excess, they also used it in the sense referred to on p. 186. Widman used it as an abbreviation for excess in 1489 (see p. 185), and Stifel in 1544 (see p. 192): by 1630 it was part of the recognized notation of algebra, and was also used as a symbol of operation.

Subtraction was indicated by Diophantus by an inverted and truncated ψ (see p. 98). The Hindoos denoted it by a dot (see p. 148). The Italian algebraists when they introduced syncopated algebra generally denoted *minus* by *M* or *m*, a line being sometimes drawn through the letter: but the practice was not uniform; Pacioli for example denoting it by *de* for *demptus* (see p. 189). The German and English algebraists introduced the present symbol which they described as *signum subtractorum*. It is most likely that the vertical bar in the symbol for plus was superimposed on the symbol for minus to distinguish the two. In origin both symbols were probably mercantile marks (see p. 185). It may be noticed that Pacioli and Tartaglia found the sign – already used to denote a division, a ratio, or a proportion indifferently (see p. 154 and p. 212). The present sign was in general use by about the year 1630, and was then employed as a symbol of operation.

Oughtred in 1631 and Harriot in 1631 both used the sign × to indicate *multiplication*: Descartes in 1637 denoted the operation by a dot. I am not aware of any symbols for it which were in previous use. Leibnitz in 1686 employed the sign ⌒ to denote multiplication, and ⌣ to denote division.

Division was generally denoted by the Arab way of writing the quantities in the form of a fraction by means of a line drawn between them in any of the forms $a - b$, a/b, or $\frac{a}{b}$. Oughtred in 1631 employed a dot to denote either division or a ratio. I do not know when the semicolon or symbol : was first introduced to denote a ratio, but it occurs in a work by Clairaut published in 1760. I believe that the current symbol for division ÷ is only a combination of the – and the :, it was

first used by Pell in 1630: this origin is made more probable by division having been sometimes formerly indicated by $\div$

The current symbol for *equality* was introduced by Recorde in 1540 (see p. 191); Xylander in 1575 denoted it by two parallel vertical lines; but in general till the year 1600 the word was written at length; and from then until the time of Newton, say about 1680, it was more frequently represented by the symbols $\propto$ or ∞ than by any other. These latter signs were used as a contraction for the first two letters of the word *aequalis*. I may add that Vieta employed the sign $=$ to denote the difference between; thus $a = b$ means with him what we denote by $a \sim b$.

The symbol :: to denote proportion, or the equality of two ratios, was introduced by Oughtred in 1631, and was brought into common use by Wallis in 1686. There is no object in having a separate symbol to express the equality of two ratios, and it is better to replace it by the sign $=$.

The sign $>$ for *is greater than* and the sign $<$ for *is less than* were introduced by Harriot in 1631, but Oughtred simultaneously invented the symbols $\sqsupset$ and $\sqsubset$ for the same purpose; and these latter continued to be generally used till the beginning of the eighteenth century, e.g. by Barrow.

The symbols $\neq$ for *is not equal to*, $\ngtr$ for *is not greater than*, and $\nless$ for *is not less than* are of quite recent introduction.

The vinculum was introduced by Vieta in 1591; and brackets were first used by Girard in 1629.

The different methods of representing the power to which a magnitude was raised have been already briefly alluded to. The earliest attempt to frame a symbolic notation was made by Bombelli in 1572 when he represented the unknown quantity by $\overset{1}{\smile}$, its square by $\overset{2}{\smile}$, its cube by $\overset{3}{\smile}$ &c. (see p. 203). In 1586 Stevinus used ①, ②, ③ &c. in a similar way; and suggested though he did not use a corresponding notation for fractional indices (see p. 204 and p. 217). In 1591 Vieta improved on this by denoting the different powers of A by A, Aq, Ac, Aqq, &c., so that he could indicate the powers of different magnitudes (see p. 205); Harriot in 1631 further

improved on Vieta's notation by writing *aa* for a^2, *aaa* for a^3, &c. (see p. 211), and this remained in use for fifty years concurrently with the index notation. Three years later, Herigone in his *Cursus mathematici* published in 2 vols. at Paris in 1634 wrote a, $a2$, $a3$,...for a, a^2, a^3....

The idea of using exponents to mark the power to which a quantity was raised—thus combining the advantages of the notations of Bombelli and of Vieta—was due to Descartes and was introduced by him in 1637; but he only used positive integral indices a^1, a^2, a^3,.... Wallis in 1659 explained the meaning of negative and fractional indices in expressions such as a^{-1}, $a^{\frac{2}{3}}$, &c. (see p. 256). The final idea of an index unrestricted in magnitude, and denoted by an expression such as a^n, was due to Newton and was introduced by him in connection with the binomial theorem (see p. 324).

There are but few special symbols in trigonometry, I may however add here the following note which contains all that I have been able to learn on the subject. The current sexagesimal division of angles is derived from the Babylonians through the Greeks. The Babylonian unit angle was the angle of an equilateral triangle; following their usual practice (see p. 6) this was divided into sixty equal parts or degrees, a degree was subdivided into sixty equal parts or minutes, and so on. The word *sine* occurs in Regiomontanus and was derived from the Arabs: the terms *secant* and *tangent* were introduced by Thomas Finck (born in Denmark in 1561 and died in 1646) in his *Geometriae rotundi*, Bâle, 1583: the word *cosecant* was (I believe) first used by Rheticus in his *Opus Palatinum*, 1596: the names *cosine* and *cotangent* were first employed by Gunther in his *Canon triangulorum*, London, 1620. The abbreviations *sin*, *tan*, *sec* were used in 1626 by Albert Girard (1590—1634), and those of *cos* and *cot* by Oughtred in 1657; but these contractions did not come into general use till Euler re-introduced them in 1748. The idea of trigonometrical *functions* was originated by John Bernoulli, and this view of the subject was elaborated in 1748 by Euler in his *Introductio in analysin* (see p. 367).

CHAPTER XIII.

THE CLOSE OF THE RENAISSANCE. 1586–1637.

SECTION 1. *The development of mechanics and experimental methods.*
SECTION 2. *Revival of interest in pure geometry.*
SECTION 3. *Mathematical knowledge at the close of the renaissance.*

THE closing years of the renaissance were marked by a revival of interest in nearly all branches of mathematics and science. As far as pure mathematics is concerned we have already seen that during the last half of the sixteenth century there had been a great advance in algebra, theory of equations, and trigonometry; and we shall shortly see (in the second section of this chapter) that in the early part of the seventeenth century some new processes in geometry were invented. If however we turn to applied mathematics it is impossible not to be struck by the fact that even as late as the middle or end of the sixteenth century no distinct progress in the theory had been made from the time of Archimedes. Statics (of solids) and hydrostatics remained in the state in which he had left them; while dynamics as a science did not exist. It was Stevinus who gave the first impulse to the renewed study of statics, and Galileo who laid the foundation of dynamics; and to their works the first section of this chapter is devoted.

The development of mechanics and experimental methods.

Simon **Stevinus*** was born at Bruges in 1548 and died at the Hague early in the seventeenth century. We know very

* An analysis of his works is given in Quetelet; see also *Notice historique sur la vie et les ouvrages de Stevinus* by J. V. Goethals, Brussels, 1841; and *Les travaux de Stevinus* by M. Steichen, Brussels, 1846.

little of his life save that he was originally a merchant's clerk at Antwerp, and at a later period of his life was the friend of Prince Maurice of Orange, by whom he was made quartermaster-general of the Dutch army.

To his contemporaries he was best known for his works on fortifications and military engineering, and the principles he laid down are said to be in accordance with those which are now usually accepted. To the general populace he was also well known on account of his invention of a carriage which was propelled by sails; this ran on the sea-shore, carried twenty-eight people, and easily outstripped horses galloping by the side: his model of it was destroyed in 1802 by the French when they invaded Holland. It was chiefly owing to the influence of Stevinus that the Dutch and French began a proper system of book-keeping in the national accounts.

I have already alluded (see p. 204) to the introduction in his *Arithmetic* published in 1586 of exponents to mark the power to which quantities were raised. For instance he wrote $3a^2 - 5a + 1$ as $3 \, ② - 5 \, ① + 1 \, ⓪$. His notation for decimal fractions was of a similar character (see p. 176). He used fractional but not negative exponents. In the same book he likewise suggested a decimal system of weights and measures.

He also published a geometry which is ingenious though it does not contain many results which were not previously known.

It is however on his *Statics and Hydrostatics* published (in Flemish) at Leyden in 1586 that his fame will rest. In this work he enunciated the triangle of forces. Till this time the science of statics had rested on the theory of the lever; but since then it has been usual to commence by proving the possibility of representing forces by straight lines, and so of reducing many theorems to geometrical propositions, and in particular to obtaining in that way a proof of the parallelogram (which is equivalent to the triangle) of forces. Stevinus also found the force which must be exerted along the line of greatest slope to support a given weight on an inclined plane—a problem the solution of which had long been in dispute. He

counting his pulse. He had been hitherto purposely kept in ignorance of mathematics, but one day by chance hearing a lecture on geometry, he was so fascinated by the science that he thenceforward devoted all his spare time to its study, and finally he got leave to discontinue his medical studies. He left the university in 1586, and almost immediately commenced his original researches.

He published in 1587 an account of the hydrostatic balance, and in 1588 an essay on the centre of gravity in solids. The fame of these works secured for him the appointment to the mathematical chair at Pisa—the stipend, as was the case with most professorships, being very small. During the next three years he carried on from the leaning tower that series of experiments on falling bodies which established the first principles of dynamics. Unfortunately the manner in which he promulgated his discoveries and the ridicule he threw on those who opposed him gave great offence, and in 1591 he was obliged to resign his position.

At this time he seems to have been much hampered by want of money. Influence was however exerted on his behalf with the Venetian senate, and he was appointed professor at Padua, a chair which he held for 18 years (1592—1610). His lectures here seem to have been chiefly on mechanics and hydrostatics, and the substance of them is contained in his treatise on mechanics which was published in 1612. In these lectures he repeated his Pisan experiments, and demonstrated that falling bodies did not (as was then believed) descend with velocities proportional amongst other things to their weights. He further shewed that if it were assumed that they descended with a uniformly accelerated motion it was possible to deduce the relations connecting velocity, space, and time which did actually exist. At a later date, by observing the times of descent of bodies sliding down inclined planes, he showed that this hypothesis was true. He also proved that the path of a projectile was a parabola, and in doing so implied the principles laid down in the first two laws of motion as enunciated by Newton. He gave an

suppose the sun the centre of the solar system was absurd, heretical, and contrary to Holy Scripture. The edict of March 5, 1616 which carried this into effect has never been repealed though it has long been tacitly ignored. It is well known that towards the middle of the seventeenth century the Jesuits evaded it by treating the theory as an hypothesis from which, if granted, certain results would follow.

In January 1632 Galileo published his dialogues on the system of the world in which in clear and forcible language he expounded the Copernican theory. In this, apparently through jealousy of Kepler's fame, he does not so much as mention Kepler's laws (the first two of which had been published in 1609 and the third in 1619) and he rejects Kepler's hypothesis that the tides are caused by the attraction of the moon. He rests the proof of the Copernican hypothesis on the absurd statement that it would cause tides because different parts of the earth would rotate with different velocities. He was more successful in showing that mechanical principles would account for the fact that a stone thrown straight up would fall again to the place from which it was thrown—a fact which had previously been one of the chief difficulties in the way of any theory which supposed the earth to be in motion.

The publication of this book was certainly contrary to the edict of 1616. Galileo was at once summoned to Rome, forced to recant, do penance, and was only released on good behaviour. The documents recently printed show that he was threatened with the torture, but that there was no intention of carrying the threat into effect.

When released he again took up his work on mechanics, and by 1636 had finished a book which was published under the title *Discorsi intorno a due nuove scienze* at Leyden in 1638. In 1637 he lost his sight, but with the aid of pupils he continued his experiments on mechanics and hydrostatics, and in particular on the possibility of using a pendulum to regulate a clock, and on the theory of impact.

An anecdote of this time has been preserved, which may

or may not be true, but is sufficiently interesting to bear repetition. It is said that he was one day interviewed by some members of one of the Florentine guilds who wanted their pumps altered so as to raise water to a height which was greater than 30 feet. Galileo thereupon made some remark upon it being desirable to first find out why the water rose at all. A bystander interfered and said there was no difficulty about that because nature abhorred a vacuum. Yes, said Galileo, but apparently it is only a vacuum which is less than 30 feet. His favorite pupil Torricelli was present, and thus had his attention directed to the question which he subsequently elucidated.

Galileo's work may I think be fairly summed up by saying that his researches on mechanics are deserving of the highest praise: his astronomical observations and his deductions therefrom were also excellent, and were expounded with a literary and scientific skill which leaves nothing to be desired; but though he produced some of the evidence which placed the Copernican theory on a satisfactory basis he did not himself make any special advance in the theory of astronomy. Galileo however will always rank as among the earliest and greatest of those who taught that science must be founded on laws obtained by experiment. An edition of his works was issued in 16 volumes by E. Albèri at Florence 1842—1856. A good many of his letters on various mathematical subjects have since been discovered, and a new and complete edition is now being prepared by Prof. Favaro of Padua for the Italian government.

The necessity of an experimental foundation for science was advocated with even greater effect by Galileo's contemporary Francis Bacon (Lord Verulam). *Francis* Bacon* was born at London on Jan. 22, 1561 and died on April 9, 1626. He was educated at Trinity College, Cambridge, 1573—1576. His career in politics and at the bar culminated in his becoming

* See his life by J. Spedding, London, 1862—74.

lord chancellor with the title of Lord Verulam: the story of his subsequent degradation for accepting bribes is well known. His great work is the *Novum organum* published in 1620 in which he lays down the principles which should guide those who are making experiments on which they propose to found a theory of any branch of physics or applied mathematics. He gave rules by which the results of induction could be tested, hasty generalizations avoided, and experiments used to check one another. The influence of this treatise in the eighteenth century was very great, but it is probable that during the preceding century it was little read, and the remark repeated by several French writers that Bacon and Descartes are the creators of modern philosophy rests on a misapprehension of his influence on his contemporaries: any detailed account of this book belongs however to the history of scientific ideas rather than to that of mathematics. The best edition of his works is that by Ellis, Spedding, and Heath in 7 volumes, London (2nd ed.), 1870.

Before leaving the subject of mechanics I may add that the two theorems known by the name of Pappus (to which I alluded on p. 93) were published by Guldinus in the fourth book of his *De centro gravitatis*, Vienna, 1635—1642. *Habakkuk* **Guldinus**, born at St Gall on June 12, 1577 and died at Grätz on Nov. 3, 1643, was of Jewish descent, but was brought up as a protestant: he was converted to Roman catholicism and became a jesuit, when he took the christian name of Paul, and it was to him that the jesuit colleges at Rome and Grätz owed their mathematical reputation. Not only were the rules in question taken without acknowledgment from Pappus, but (according to Montucla) the proof of them given by Guldinus was faulty, though he was successful in applying them to the determination of the volumes and surfaces of certain solids. The theorems were however previously unknown, and their enunciation excited great interest.

Revival of interest in pure geometry.

The close of the sixteenth century was marked not only by the attempt to found a theory of dynamics based on laws derived from experiment, but also by a revived interest in geometry. This was largely due to the influence of Kepler.

Johann Kepler*, one of the founders of modern astronomy, was born of humble parents near Stuttgart on Dec. 27, 1571 and died at Ratisbon on Nov. 15, 1630. He was educated at Tübingen; in 1593 he was appointed professor at Grätz, where he made the acquaintance of a wealthy and beautiful widow whom he married, but found too late that he had purchased his freedom from pecuniary troubles at the expense of domestic happiness. In 1599 he accepted an appointment as assistant to Tycho Brahe, and in 1601 succeeded his master as astronomer to the emperor Rudolph II. But his career was dogged by bad luck; first his stipend was not paid; next his wife went mad and then died; and though he married again in 1611 this proved an even more unfortunate venture than before, for though to secure a better choice he took the precaution to make a preliminary selection of eleven girls whose merits and demerits he carefully analysed in a paper which is still extant, he finally selected a wrong one; while to complete his discomfort he was expelled from his chair, and narrowly escaped condemnation for heterodoxy. During all this time he depended for his income on telling fortunes and casting horoscopes, for as he says "nature which has conferred upon every animal the means of existence has designed astrology as an adjunct and ally to astronomy." He seems however to have had no scruple in charging heavily for his services, and to the surprise of his contemporaries was found at his death to have a considerable hoard of money. He died while on a journey to try and recover for the benefit of his children some of the arrears of his stipend.

* See *Johann Keppler's Leben und Wirken,* by J. L. E. Breitschwert, Stuttgart, 1831; and R. Wolf's *Geschichte der Astronomie,* Munich, 1871.

In describing Galileo's work I alluded briefly to the three laws in astronomy that Kepler had discovered, and in connection with which his name will always be associated; and I have already mentioned the prominent part he took in bringing logarithms into general use on the continent. These are familiar facts, but it is not so generally known that he was also a geometrician and algebraist of considerable power; and that he, Desargues, and perhaps Galileo may be considered as forming a connecting link between the mathematicians of the renaissance and those of modern times.

Kepler's work in geometry consists rather in certain general principles which he laid down and illustrated by a few cases than in any systematic exposition of the subject. In a short chapter (chap. IV.) on conics inserted in his *Paralipomena* published in 1604 he lays down what has been called the principle of continuity; and gives as an example the statement that a parabola is at once the limiting case of an ellipse or of a hyperbola; he illustrates the same doctrine by reference to the foci of conics (the word *focus* was introduced by him); and he also explains that parallel lines should be regarded as meeting at infinity.

In his *Stereometry* which was published in 1615 he determines the volumes of certain vessels and the areas of certain surfaces by means of infinitesimals, instead of by the long and tedious method of exhaustions. These investigations as well as those of 1604 arose from a dispute with a wine merchant as to the proper way of gauging the contents of a cask. This use of infinitesimals was objected to by Guldinus and other writers as inaccurate, but though the methods of Kepler are not altogether free from objection he was substantially correct, and by applying the law of continuity to infinitesimals he prepared the way for Cavalieri's method of indivisibles and the infinitesimal calculus of Newton and Leibnitz.

Kepler's work on astronomy lies outside the scope of this book. I will only mention that it was founded on the observations of Tycho Brahe (born at Knudstrup in 1546 and

died at Prague in 1601) whose assistant he was. His three laws of planetary motion were the result of many and laborious efforts to reduce the phenomena of the solar system to certain simple rules. The first two were published in 1609, and stated that the planets described ellipses round the sun, the sun being in the focus; and that the line joining the sun to any planet swept over equal areas in equal times. The third was published in 1619, and stated that the squares of the periodic times of the planets were proportional to the cubes of the major axes of their orbits. I ought also to add that he attempted to explain why these motions took place by a hypothesis which is not very different from Descartes' theory of vortices.

A complete edition of Kepler's works was published by C. Frisch at Frankfort in 8 volumes 1858—71; and an analysis of the mathematical part of his chief work, the *Harmonice mundi*, is given by Chasles in his *Aperçu historique*.

While the conceptions of the geometry of the Greeks were being extended by Kepler, a Frenchman, whose name until recently was almost unknown, was inventing a new method of investigating the subject—a method which is now known as projective geometry. This was the discovery of Desargues whom I put (with some hesitation) at the close of this period, and not among the mathematicians of modern times.

Gerard Desargues*, born at Lyons in 1593 and died in 1662, was by profession an engineer and architect, but he gave some courses of gratuitous lectures in Paris from 1626 to about 1630 which made a great impression on his contemporaries. Both Descartes and Pascal had a very high opinion of his work and abilities, and both made considerable use of the theorems he had enunciated.

Most of his researches were embodied in his *Brouillon projet des coniques* published in 1639 a copy of which was discovered by Chasles in 1845. I take the following summary

* See *La vie et les œuvres de Desargues* by M. Poudra, 2 vols., Paris, 1864.

of it from a recent work on geometry. Desargues commences with a statement of the doctrine of infinity as laid down by Kepler: thus the points at the opposite ends of a straight line are regarded as coincident, parallel lines are treated as meeting at a point at infinity, and parallel planes on a line at infinity, while a straight line may be considered as a circle whose centre is at infinity. The theory of involution of six points, with its special cases, is fully laid down, and the projective property of pencils in involution is established. The theory of polar lines is expounded, and its analogue in space suggested. A tangent is defined as the limiting case of a secant, and an asymptote as a tangent at infinity. He shows that the lines which join four points in a plane determine three pairs of lines in involution on any transversal, and from any conic through the four points another pair of lines can be obtained which are in involution with any two of the former. He proves that the points of intersection of the diagonals and the two pairs of opposite sides of any quadrilateral inscribed in a conic are a conjugate triad with respect to the conic, and when one of the three points is at infinity its polar is a diameter; but he fails to explain the case in which the quadrilateral is a parallelogram, although he had framed the conception of a straight line which was wholly at infinity. The book therefore may be fairly said to contain the fundamental theorems on involution, homology, poles and polars, and on the relations between two conics in perspective.

The influence exerted by the lectures of Desargues on Descartes, Pascal, and the French geometricians of the seventeenth century was considerable; but the subject of projective geometry soon fell into oblivion, chiefly because the analytical geometry of Descartes was so much more powerful as a method of proof or discovery.

The researches of Kepler and Desargues will serve to remind us that, as the geometry of the Greeks was not capable of much further extension, mathematicians were now beginning to seek for new methods of investigation, and were extending

the conceptions of geometry. The invention of analytical geometry and of the infinitesimal calculus temporarily diverted attention from pure geometry, but at the beginning of the present century there was a revival of interest in it, and since then it has been a favorite subject of study with many mathematicians.

Mathematical knowledge at the close of the renaissance.

Thus by the beginning of the seventeenth century we may say that the fundamental principles of arithmetic, algebra, theory of equations, and trigonometry had been laid down, and the outlines of the subjects as we know them had been traced. It must however be remembered that there were no good elementary text-books on these subjects; and a knowledge of them was thus confined to those who could extract it from the ponderous treatises in which it lay buried. Though much of the modern algebraical and trigonometrical notation had been introduced it was not familiar to mathematicians nor was it even universally accepted; and it was not until the end of the seventeenth century that the language of the subject was definitely fixed. Considering the absence of good text-books I am inclined rather to admire the rapidity with which it came into universal use, than to cavil at the hesitation which writers who desired to make their results perfectly clear showed to trust to it alone.

If we turn to applied mathematics we find on the other hand that the science of statics had made but little advance in the eighteen centuries that had elapsed since the time of Archimedes, while the foundations of dynamics were only laid by Galileo at the close of the sixteenth century. In fact as we shall see later it was not until the time of Newton that the science of mechanics was placed on a satisfactory basis. The fundamental conceptions of mechanics are difficult, but the ignorance of the principles of the subject shown by the mathe-

maticians of this time is greater than would have been anticipated from their knowledge of pure mathematics.

With this exception we may say that the principles of analytical geometry and of the infinitesimal calculus were needed before there was likely to be much further progress. The former was employed by Descartes in 1637, the latter was invented by Newton (and possibly independently by Leibnitz) some thirty or forty years later: and their introduction may be taken as marking the commencement of the period of modern mathematics.

THIRD PERIOD.

MODERN MATHEMATICS.

This period begins with the invention of analytical geometry and the infinitesimal calculus. The mathematics is far more complex than that produced in either of the preceding periods: but it may be generally described as characterized by the development of analysis, and its application to the phenomena of nature.

I continue the chronological arrangement of the subject. Chapter XV. contains the history of the forty years from 1635 to 1675, and an account of the mathematical discoveries of Descartes, Cavalieri, Pascal, Wallis, Fermat, and Huygens. Chapter XVI. is given up to a discussion of Newton's researches. Chapter XVII. contains an account of the works of Leibnitz and his followers during the first half of the eighteenth century (including d'Alembert), and also of the contemporary English school to the death of Maclaurin. The works of Euler, Lagrange, Laplace, and their contemporaries form the subject-matter of chapter XVIII. Lastly in chapter XIX. I have added some notes on a few of the mathematicians of recent times; but I exclude living writers, and partly because of this, partly for other reasons there given, the account of contemporary mathematics does not profess to be exhaustive or complete. I should add that the division between the chapters is not so well defined as I should have wished, and the lives of the mathematicians considered at the end of one chapter generally overlap the lives of some of those who form the subject-matter of the next chapter.

CHAPTER XIV.

FEATURES OF MODERN MATHEMATICS.

The division between this period and that treated in the last six chapters is by no means so well defined as that which separates the history of Greek mathematics from the mathematics of the middle ages. The methods of analysis used in the seventeenth century and the kind of problems attacked changed but gradually; and the mathematicians at the beginning of this period were in immediate relations with those at the end of that last considered. For this reason some writers have divided the history of mathematics into two parts only, treating the schoolmen as the lineal successors of the Greek mathematicians, and dating the creation of modern mathematics from the introduction of the Arab text-books into Europe. The division I have given is I think more convenient, for the introduction of analytical geometry and of the calculus completely revolutionized the development of the subject, and it therefore seems preferable to take their invention as marking the commencement of modern mathematics.

The time that has elapsed since these methods were invented has been a period of incessant intellectual activity in all departments of knowledge, and the progress made in mathematics has been immense. The greatly extended range of knowledge and the rapid intercommunication of ideas due to printing increase the difficulties of a historian; while the mass of materials which has to be mastered, the absence of perspective, and even the echoes of old controversies combine to make it very difficult to give a clear and just account of the

development of the subject. As however the leading facts are generally known, and the works published during this time are accessible to any student, I may deal more concisely with the lives and writings of modern mathematicians than with those of their predecessors.

Roughly speaking we may say that five distinct stages in the history of this period can be discerned.

First of all there is the invention of analytical geometry by Descartes in 1637; and almost at the same time the introduction of the method of indivisibles, by the use of which areas, volumes, and the positions of centres of mass can be determined by summation in a manner analogous to that effected now-a-days by the aid of the integral calculus. The method of indivisibles was soon superseded by the integral calculus. Analytical geometry however maintains its position as part of the necessary training of every mathematician, and is incomparably more potent than the geometry of the ancients for all purposes of research. The latter is still no doubt an admirable intellectual training, and it frequently affords an elegant demonstration of some proposition the truth of which is already known, but it requires a special procedure for every problem attacked. The former on the other hand lays down a few simple rules by which any property can be at once proved or disproved.

In the *second* place we have the invention of the fluxional or differential calculus about 1666 (and possibly an independent invention of it in 1674). Wherever a quantity changes according to some continuous law (and most things in nature do so change) the differential calculus enables us to measure its rate of increase or decrease; and from its rate of increase or decrease the integral calculus enables us to find the original quantity. Formerly every separate function of x such as $(1+x)^n$, $\log(1+x)$, $\sin x$, $\tan^{-1} x$, &c. could only be expanded in ascending powers of x by means of such special procedure as was suitable for that particular problem; but by the aid of the calculus the expansion of any function of x in ascending

powers of x is in general reducible to a single simple rule which covers all cases alike. So again the theory of maxima and minima, the determination of the lengths of curves or the areas enclosed by them, the determination of surfaces, of volumes, and of centres of mass, and many other problems are each reducible to a single rule. The theories of differential equations, of the calculus of variations, of finite differences, &c. are the developments of the ideas of the calculus.

These subjects were the two great instruments of further progress. In both of them a sort of machine has been constructed: to solve a problem, it is only necessary to put in the particular function operated on, or the equation of the particular curve or surface to be considered, and on performing certain simple processes the result comes out. The validity of the process is proved once for all, and it is no longer requisite to invent some special method for every separate function, curve, or surface.

In the *third* place Huygens laid the foundation of a satisfactory treatment of dynamics, and Newton reduced it to an exact science. The latter mathematician proceeded to apply these two new engines of analysis not only to numerous problems in the mechanics of solids and fluids on the earth but to the solar system: the whole of mechanics terrestrial and celestial was thus brought within the domain of mathematics. There is no doubt that Newton used the calculus to obtain many of his results, but he seems to have thought that if his demonstrations were established by the aid of a new science which was at that time generally unknown, his critics (who would not understand the fluxional calculus) would fail to realize the truth and importance of his discoveries. He therefore determined to give geometrical proofs of all his results. He accordingly cast the *Principia* into a geometrical form, and thus presented it to the world in a language which all men could then understand. The theory of mechanics was completed by Laplace and Lagrange towards the end of the eighteenth century.

In the *fourth* place we may say that during this period the chief branches of physics have been brought within the scope of mathematics. This extension of the domain of mathematics was commenced by Huygens and Newton when they propounded their theories of light; but it was not until the beginning of this century that sufficiently accurate observations were made in most physical subjects to enable mathematical reasoning to be applied to them. From the results of the observations and experiments which have been since published numerous and far-reaching conclusions have been obtained by the use of mathematics, but we now want some more simple hypotheses from which we can deduce those laws which at present form our starting-point. If, to take one example, we could say in what electricity consisted we might get some simple laws or hypotheses from which by the aid of mathematics all the observed phenomena could be deduced; in the same way as Newton deduced all the results of physical astronomy from the law of gravitation. All lines of research seem moreover to indicate that there is an intimate connection between the different branches of physics, e.g. between light, heat, electricity, and magnetism. The ultimate explanation of this and of the leading facts in physics seems to demand a study of molecular physics; a knowledge of molecular physics in its turn seems to require some theory as to the constitution of matter; it would further appear that the key to the constitution of matter is to be found in chemistry or chemical physics. So the matter stands at present. Helmholtz in Germany, and Maxwell and Sir William Thomson in England, have done a great deal in applying mathematics to physics; but the connection between the different branches of physics, and the fundamental laws of those branches (if there are any simple ones), are riddles which are yet unsolved. This history does not pretend to deal with problems which are now the subject of investigation, and though mathematical physics forms a large part of "modern mathematics" I shall not treat of it in any detail.

Fifthly this period has seen an immense extension of pure mathematics. Much of this is the creation of comparatively recent times, and I regard the details of it as outside the limits of this book, though in chapter XIX. I have allowed myself to enumerate the subjects discussed. The most characteristic features of it are the development of higher geometry, of higher arithmetic (i.e. the theory of numbers), of higher algebra (including the theory of forms), and the discussion of functions of double and multiple periodicity.

CHAPTER XV.

HISTORY OF MATHEMATICS FROM DESCARTES TO HUYGENS. CIRC. 1635—1675.

I propose in this chapter to consider the history of mathematics during the forty years in the middle of the seventeenth century. I regard Descartes, Cavalieri, Pascal, Wallis, Fermat, possibly Barrow, and Huygens as the leading mathematicians of this time. I shall treat them in that order, and I shall conclude with a brief list of the more eminent remaining mathematicians of the same date.

I have already stated that the mathematicians of this period—and the remark applies more particularly to Descartes, Pascal, and Fermat—were largely influenced by the teaching of Kepler and Desargues, and I would repeat again that I regard these latter and Galileo as forming a connecting link between the writers of the renaissance and those of modern times. I should also add that the mathematicians considered in this chapter were contemporaries, and although I have tried to place them roughly in such an order that their chief works shall come in a chronological arrangement it is essential to remember that they were in relation one with the other, and in general acquainted with one another's researches as soon as these were published.

Subject to these remarks we may consider Descartes as the first of the modern school of mathematics. *René* Descartes* was born near Tours on March 31, 1596 and died at Stock-

* See *La vie de Descartes* by Baillet, 2 vols., Paris, 1691, which is summarized in K. Fischer's *Geschichte der Neuern Philosophie*, Mannheim, 1865. A tolerably complete account of his mathematical and physical investigations is given in Ersch and Gruber's *Encyclopädie*, and is the authority for most of the statements here contained.

holm on Feb. 11, 1650; he was thus a contemporary of Galileo and Desargues. His father, who as the name implies was of a good family, was accustomed to spend half the year at Rennes when the local parliament in which he held a commission as councillor was in session, and the rest of the time on his family estate of *les Cartes* at la Haye. René, the second of a family of two sons and one daughter, was sent at the age of eight years to the Jesuit School at la Flèche, and of the admirable discipline and education there given he speaks most highly. On account of his delicate health he was permitted to lie in bed till late in the mornings; this was a custom which he always followed, and when he visited Pascal in 1647 he told him that the only way to do good work in mathematics and to preserve his health was never to allow anyone to make him get up in the morning before he felt inclined to do so; an opinion which I chronicle for the benefit of any schoolboy into whose hands this work may fall.

On leaving school in 1612 Descartes went to Paris to be introduced to the world of fashion. Here through the medium of the Jesuits he made the acquaintance of *Mydorge** and renewed his schoolboy friendship with Father *Mersenne*†, and together with them he devoted the two years of 1615 and 1616 to the study of mathematics. At that time the army and the church were the only careers open to a man of position,

* *Claude Mydorge*, born at Paris in 1585 and died in 1647, belonged to a distinguished "family of the robe," and was himself a councillor at Châtelet, and then treasurer to the local parliament at Amiens. He published some works on optics of which one issued in 1631 is still extant, and a treatise on conic sections in 1641. He also left a manuscript containing solutions of over a thousand geometrical problems, most of which are said to be very ingenious; the enunciations were published by M. Charles Henry in 1882.

† *Marin Mersenne*, born in 1588 and died at Paris in 1648, was a Franciscan friar, who made it his business to know and correspond with all the French mathematicians of that date and many of their foreign contemporaries. He published a translation of Galileo's mechanics in 1634; and he wrote a synopsis of mathematics which was printed in 1664. He has also left an account of some experiments in physics.

and in 1617 Descartes joined the army of Prince Maurice of Orange then at Breda. Walking through the streets he saw a placard in Dutch which excited his curiosity, and stopping the first passer asked him to translate it into either French or Latin. The stranger, who happened to be Isaac Beeckman, the head of the Dutch College at Dort, offered to do so if Descartes would answer it: the placard being in fact a challenge to all the world to solve a geometrical problem therein given. Descartes worked it out within a few hours, and a warm friendship between him and Beeckman was the result. This unexpected test of his mathematical attainments made the uncongenial life of the army distasteful to him, but under family influence and tradition he remained a soldier, and was persuaded at the commencement of the thirty years' war to volunteer in the army of Bavaria. He continued all this time to occupy his leisure with mathematical studies, and was accustomed to date the first ideas of his new philosophy and of his analytical geometry from three dreams which he experienced on the night of Nov. 10, 1619 at Neuberg when campaigning on the Danube. He always regarded this as the critical day of his life, and one which determined his whole future.

He resigned his commission in the spring of 1621, and spent the next five years in travel, during most of which time he continued to study pure mathematics; he then settled at Paris, and for two years interested himself in the construction of optical instruments. But these pursuits were only the relaxations of one who failed to find in philosophy that theory of the universe which he was convinced finally awaited him. In 1628 Cardinal de Berulle, the founder of the Oratorians, met Descartes, and was so much impressed by his conversation that he urged on him the duty of devoting his life to the examination of truth. Descartes agreed, and the better to secure himself from interruption moved to Holland then at the height of its power. There for twenty years he lived, giving up all his time to philosophy and mathematics. Science, he says, may be compared to a tree, metaphysics is the root,

physics is the trunk, and the three chief branches are mechanics, medicine, and morals, these forming the three applications of our knowledge, namely to the external world, to the human body, and to the conduct of life: and with these subjects alone his writings are concerned. He spent the first four years, 1629 to 1633, of his stay in Holland in writing *Le monde* which is a physical theory of the universe; but finding that its publication was likely to bring on him the hostility of the church, and having no desire to pose as a martyr, he abandoned it: the incomplete manuscript was published in 1664. He then devoted himself to composing a treatise on universal science (including and illustrated by dioptrics, meteors and geometry); this was published at Leyden in 1637 under the title *Discours de la méthode pour bien conduire sa raison et chercher la vérité dans les sciences*, and from it dates the invention of analytical geometry. In 1641 he published a work called *Meditations* in which he explained at greater length his views of philosophy as sketched out in the *Discours*. In 1644 he issued the *Principia philosophiae*, the greater part of which was devoted to physical science, especially the laws of motion and the theory of vortices. In 1647 he received a pension from the French court in honour of his discoveries. He went to Sweden on the invitation of the Queen in 1649, and died a few months later of inflammation of the lungs.

In appearance, Descartes was a small man with large head, projecting brow, prominent nose, and black hair coming down to his eyebrows. His voice was feeble. Considering the range of his studies he was by no means widely read, and he despised both learning and art unless something tangible could be extracted therefrom. In disposition he was cold and selfish. It is said that he remarked that nearly every man above forty if married heartily regretted the fetters he had imposed on himself, while if single he complained of his loneliness: thus in either case the result was disappointment, and as no preliminary experiment was possible all that a wise man could do was to judge which course was likely to prove the least evil in

section of the *Discours*. This is divided into three books; the first two of these treat of analytical geometry, and the third is an analysis of the algebra then current. It is throughout very difficult to follow the reasoning, but the obscurity was intentional and due to the jealousy of Descartes. "Je n'ai rien omis," says he, "qu'à dessein..... j'avois prévu que certaines gens qui se vantent de sçavoir tout n'auroient pas manqué de dire que je n'avois rien écrit qu'ils n'eussent sçu auparavant si je me fusse rendu assez intelligible pour eux." The sixth commences with an analytical solution of a certain problem which had been propounded by Pappus in the seventh book of his Συναγωγή and of which some particular cases had been considered by Euclid and Apollonius. The general theorem had baffled previous geometricians, and it was in the attempt to solve it that Descartes was led to the invention of analytical geometry. The full enunciation of the problem is rather involved, but the most important case is to find the locus of a point such that the product of the perpendiculars on m given straight lines shall be in a constant ratio to the product of the perpendiculars on n other given straight lines. The ancients had solved this for the case $m = 1$, $n = 1$, and the case $m = 1$, $n = 2$; and Pappus had further stated that if $m = n = 2$ the locus was a conic, but gave no proof. Descartes also failed to prove this by pure geometry, but in his efforts to do so he introduced a system of coordinates, and thus shewed that the curve was represented by an equation of the second degree, that is was a conic. Newton subsequently gave an elegant solution of the problem by pure geometry.

Descartes divided curves into two classes; namely, geometrical and mechanical curves. He defined geometrical curves as those which can be generated by the intersection of two lines each moving parallel to one coordinate axis with "commensurable" velocities, by which he meant that $\frac{dy}{dx}$ was an algebraic function, e.g. the ellipse or the cissoid. He called a curve mechanical when the ratio of the velocities of these lines w

"incommensurable," by which he meant that $\frac{dy}{dx}$ was a transcendental function, e.g. the cycloid or the quadratrix. Descartes confined his discussion to the application of algebra to geometry, and did not treat of the theory of mechanical curves. The classification into algebraical and transcendental curves now usual is due to Newton (see p. 322).

In this work Descartes paid particular attention to the theory of the tangents to curves—as might perhaps be inferred from his system of classification just alluded to. The then current definition of a tangent at a point was a straight line through the point such that between it and the curve no other straight line could be drawn, i.e. the straight line of closest contact. Descartes proposed to substitute for this that the tangent was the limiting position of the secant; Fermat, and at a later date Maclaurin and Lagrange, adopted this definition. Barrow, followed by Newton and Leibnitz, considered a curve as the limit of an inscribed polygon when the sides become indefinitely small, and stated that a side of the polygon when produced became in the limit a tangent to the curve. Roberval on the other hand defined a tangent at a point as the direction of motion at that instant of a point which was describing the curve. The results are the same whichever definition is selected, but the controversy as to which definition was the correct one was none the less lively. Descartes illustrated his theory by giving the general rule for drawing tangents to a roulette.

The method used by Descartes to find the tangent at any point of a given curve was substantially as follows. He determined the centre and radius of a circle which should cut the curve in two consecutive points there. The tangent to the circle at that point will be the required tangent to the curve. In modern text-books it is usual to express the condition that two of the points in which a straight line (such as $y = mx + c$) cuts the curve shall coincide with the given point: this enables us to determine m and c, and so the equation of the tangent

them is at once found. Descartes however did not ve to do this, but selecting a circle as the simplest curve one to which he knew how to draw a tangent, he so fixe circle as to make it touch the given curve and thus re the problem to drawing a tangent to a circle. I should in passing that he only applied this method to curves w are symmetrical about an axis, and he took the centre o circle on the axis. As an illustration of this method I consider the case in which it is required to draw a tange the parabola $y^2 = 4ax$ at the point (x', y'). Consider the whose centre is at $(h, 0)$ and radius r: its equation is

$$(x-h)^2 + y^2 = r^2.$$

The abscissae of the points where the circle and parabol are given by the equation

$$(x-h)^2 + 4ax = r^2.$$

The roots of this will be equal if $4ah = 4a^2 + r^2$, and the of each root will then be $h-2a$. Hence $h = 2a + x'$ i.e. $h = a$ and $r^2 = 4a(x'+a)$. This circle will now have the same gent at the point (x', y') as the parabola has, and as the is determined and the tangent to a circle can always be d the problem is solved.

The third book of the treatise on geometry is an analy the *algebra* then current, and it largely affected the lang of the subject by fixing the custom of employing the lette the beginning of the alphabet to denote known quantitie those at the end of the alphabet to denote unknown quan and by introducing the system of indices now in us think Descartes was the first to realize that his letters m represent any quantities, positive or negative, and that i sufficient to prove a proposition for one general case pare the old procedure as illustrated on p. 201). In thi he enunciated the rule for determining a limit to the nu of positive and of negative roots of an algebraical equa which is still known by his name; and introduced the m of indeterminate coefficients for the solution of equations.

believed that he had given a method by which algebraical equations of any order could be solved, but in this he was mistaken.

The first section of the *Discours* was devoted to *optics*. The chief interest of this consists in the statement given of the law of refraction. This appears to have been taken from Snell's work (see foot-note p. 218), but not only is there no acknowledgment of the source from which it was obtained, but it is enunciated in such a way as to lead a careless reader to suppose that it is due to the researches of Descartes. Descartes would seem to have repeated Snell's experiments when in Paris in 1626 or 1627, and it is possible that he subsequently forgot how much he owed to Snell's earlier investigations. A large part of the optics is devoted to determining the best shape of the lenses for a telescope, but the mechanical difficulties in grinding surfaces of glass to a required form are so great as to render these investigations of little practical use. Descartes seems to have been doubtful whether to regard the rays of light as proceeding from the eye and so to speak touching the object, as the Greeks had done, or as proceeding from the object and so affecting the eye; but since he considered the velocity of light to be infinite he did not deem the point particularly important.

In the second section of the *Discours*, entitled *meteors*, he explained the rainbow. He was however unacquainted with the unequal refrangibility of rays of light of different colours and the explanation is therefore incomplete.

Fermat criticised the methods used in the *Discours* in a letter addressed to a mutual friend who shewed it to Descartes. The latter was very wroth at these comments, and in turn bitterly attacked Fermat's work on maxima and minima. The obscurity of both writers was a chief cause of the quarrel, and it would be unnecessary now to mention it, were it not for the large part it occupies in the scientific history of the time.

Descartes' physical theory of the universe, embodying most of the results contained in his earlier and unpublished *Le*

monde, was given in his *Principia*, 1644, and rests on a metaphysical basis. He commences with a discussion on motion; and then lays down ten laws of nature, of which the first two are almost identical with the first two laws of motion as given by Newton (see pp. 310, 311); the remaining eight laws are inaccurate. He next proceeds to discuss the nature of matter which he regards as uniform in kind, though there are three forms of it. He assumes that the matter of the universe must be in motion, and that the motion must result in a number of vortices. He states that the sun is the centre of an immense whirlpool of this matter, in which the planets float and are swept round like straws in a whirlpool of water. Each planet is supposed to be the centre of a secondary whirlpool by which its satellites are carried: these secondary whirlpools are supposed to produce variations of density in the surrounding medium which constitute the primary whirlpool, and so cause the planets to move in ellipses and not in circles. All these assumptions are quite arbitrary and are unsupported by any investigation. It is not difficult to prove that on his hypotheses the sun would be in the centre of these ellipses and not at a focus (as Kepler had shewn was the case), and that the weight of a body at every place on the surface of the earth except the equator would act in a direction which was not vertical. It will however be sufficient here to say that Newton in the second book of his *Principia*, 1687, considered the theory in detail, and showed that its consequences are not only inconsistent with each of Kepler's laws and with the fundamental laws of mechanics, but are also at variance with the ten laws of nature assumed by Descartes (see p. 319). Still, in spite of its crudeness and its inherent defects, the theory of vortices marks a fresh era in astronomy, for it was an attempt to explain the phenomena of the whole universe by the same mechanical laws which experiment shews to be true on the earth.

Almost contemporaneously with the publication in 1637 of Descartes' geometry, the principles of the integral calculus, so

far as they are concerned with summation, were being worked out in Italy. This was effected by what was called the principle of indivisibles, and was the invention of Cavalieri. It was applied to numerous problems connected with the quadrature of curves and surfaces, the determination of volumes, and the positions of centres of mass to the complete exclusion of the tedious method of exhaustions used by the Greeks. In principle the methods are the same, but the notation of indivisibles is much more concise and convenient. It was in its turn superseded at the beginning of the eighteenth century by the integral calculus, but its use will be familiar to all mathematicians who have read any commentary on the first section of the first book of Newton's *Principia* in the application of lemmas 2 and 3 to the determination of areas, volumes, &c.

Bonaventura **Cavalieri*** was born at Milan in 1598 and died at Bologna in 1647. He became a Jesuit at an early age; on the recommendation of the Order he was in 1629 made professor of mathematics at Bologna; and he continued to occupy the chair there until his death.

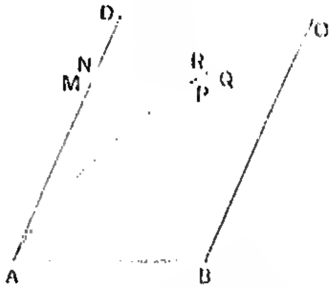
The principle of indivisibles had been used by Kepler (see p. 225) in 1604 and 1615 in a somewhat crude form. It was first stated by Cavalieri in 1629, but he did not publish his results till 1635. In his early enunciation of the principle in 1635 Cavalieri asserted that a line was made up of an infinite number of points (each without magnitude), a surface of an infinite number of lines (each without breadth), and a volume of an infinite number of surfaces (each without thickness). To meet the objections of Guldinus and others the statement was recast, and in its final form as used by the mathematicians of the seventeenth century it was published in Cavalieri's *Exercitationes geometricae sex* in 1647. These contain the first rigid demonstration of the properties of Pappus (see p. 93 and

* Cavalieri's life has been written by P. Frisi, Milan, 1778; and by F. Predari, Milan, 1843. The leading facts are given in the *éloge* on him by Gabrio Piola, Milan, 1844.

p. 223). Cavalieri's works on the subject were reissued with his later corrections in 1653.

The method of indivisibles is simply that any magnitude may be divided into an infinite number of small quantities which can be made to bear any required ratios (e.g. equality) one to the other. The analysis given by Cavalieri is very involved (chiefly in consequence of his limited powers of summation) and is not worth quoting. The principle is illustrated by the following example, the substance of which is taken from Wallis.

Let it be required to find the area bounded by the parabola APC the tangent at A, and any diameter DC. Complete the



parallelogram $ABCD$. Divide AD into n equal parts, let AM contain r of them, and let MN be the $(r+1)$th part. Draw MP and NQ parallel to AB, and draw PR parallel to AD. Then when n becomes indefinitely large the curvilinear area $APCD$ will be the limit of the sum of all parallelograms like PN. Now

$$\text{area } PN : \text{area } BD = MP \,.\, MN : DC \,.\, AD.$$

But by the properties of the parabola

$$MP : DC = AM^2 : AD^2 = r^2 : n^2,$$

and $$MN : AD = 1 : n.$$

Hence $$MP \,.\, MN : DC \,.\, AD = r^2 : n^3.$$

That is $$\text{area } PN : \text{area } BD = r^2 : n^3.$$

Therefore ultimately

$$\text{area } APCD : \text{area } BD = 1^2 + 2^2 + \dots + (n-1)^2 : n^3$$
$$= \tfrac{1}{6} n (n-1)(2n-1) : n^3$$

which in the limit $= 1 : 3$.

It is perhaps worth noticing that Cavalieri and his successors always used the method to find the ratios of two areas, volumes, or magnitudes of the same kind and dimensions: that is they never thought of an area as containing so many units of area. The idea of comparing a magnitude with a unit of the same kind seems to have been due to Wallis.

It is evident that in its direct form the method is only applicable to a few curves. I should add that in the case where m is a positive integer Cavalieri proved that the limit, when n is infinite, of $\dfrac{1^m + 2^m + \dots + n^m}{n^{m+1}}$ is $\dfrac{1}{m+1}$, i.e. he found the value of $\int_0^1 x^m dx$.

Among the contemporaries of Descartes none displayed greater natural genius than Pascal, but his reputation rests more on what he might have done than on what he actually effected, as during a considerable part of his life he deemed it his duty to devote his whole time to religious exercises.

Blaise **Pascal** was born at Clermont on June 19, 1623 and died at Paris on Aug. 19, 1662. His father, a local judge at Clermont and himself of some scientific reputation, moved to Paris in 1631 partly to prosecute his own scientific studies, partly to carry on the education of his only son who had already displayed extraordinary ability. Pascal was kept at home in order to ensure his not being overworked, and with the same object it was directed that his education should at first be confined to the study of languages and should not include any mathematics. This naturally excited the boy's curiosity, and one day being then twelve years old he asked in what geometry consisted. His tutor replied that it was the science of con-

structing exact figures and of determining the proportions between their different parts. Pascal, stimulated no doubt by the injunction against reading it, gave up his play-time to this new study, and in a few weeks had discovered for himself many properties of figures, and in particular the proposition that the sum of the angles of a triangle is equal to two right angles. I have read somewhere, but I cannot lay my hand on the authority, that his proof merely consisted in turning the angular points of a triangular piece of paper over so as to meet in the centre of the inscribed circle. A similar demonstration can be got by turning the angular points over so as to meet at the foot of the perpendicular drawn from the biggest angle to the opposite side. His father struck by this display of ability gave him a copy of Euclid's *Elements*, a book which Pascal read with avidity and soon mastered.

At the age of 14 he was admitted to the weekly meetings of Roberval, Mersenne, Mydorge, and other French geometricians from which (in 1666) the French Academy sprung*. At 16 he wrote an essay on conic sections; and in 1641 at the age of 18 he constructed the first arithmetical machine, a machine which was improved and patented a few years later, in 1649. His correspondence with Fermat about this time shews that he was then turning his attention to analytical geometry and physics. He repeated Torricelli's experiments, by which the pressure of the atmosphere could be weighed, and confirmed the theory of the barometer by obtaining at the same instant its reading at different altitudes on the hill of Puy-de-Dôme.

In 1650, when in the midst of these researches, his older sister died, and Pascal suddenly abandoned his favourite pursuits to study religion, or as he says in his *Pensées* "to contemplate the greatness and the misery of man"; and about the same time he persuaded his younger and only other sister to enter the Port Royal society. Such at least is the account of his retirement which has been generally accepted as correct,

* The French Academy was created by ordinance of Louis XIV. on Dec. 22, 1666.

but some recent writers have doubted whether he did then so completely withdraw from the world.

In 1653 he had to administer his father's estate. He now took up his old life again, and made several experiments on the pressure of gases and liquids; it was also about this period that he invented the arithmetical triangle, and together with Fermat created the calculus of probabilities. He was meditating marriage when an accident again turned the current of his thoughts to a religious life. He was driving a four-in-hand on Nov. 23, 1654, when the horses ran away; the two leaders dashed over the parapet of the bridge at Neuilly, and Pascal was only saved by the traces breaking. Always somewhat of a mystic, he considered this a special summons to abandon the world. He wrote an account of the accident on a small piece of parchment, which for the rest of his life he wore next to his heart to perpetually remind him of his covenant; and shortly moved to Port Royal where he continued to live until his death in 1662.

His famous *Provincial Letters* directed against the Jesuits, and his *Pensées*, were written at this time, and are the first example of that finished form which is characteristic of the best French literature. The only mathematical work that he did after retiring to Port Royal was the essay on the cycloid in 1658. He was suffering from insomnia and tooth-ache when the idea occurred to him, and to his surprise his teeth immediately ceased to ache. Regarding this as a divine intimation to proceed with the problem, he worked incessantly for eight days at it, and completed a tolerably full account of the geometry of the cycloid. Always delicate, he had injured his health by his incessant study; from the age of 17 or 18 he suffered terribly from insomnia and acute dyspepsia; and at the time of his death he was completely worn out.

A complete edition of his numerous pamphlets and his correspondence was published by Lahure in 3 vols. at Paris in 1858.

I now proceed to consider his mathematical works in rather greater detail.

His early essay on the *Geometry of conics*, written in 1639 but not published till 1779, seems to have been founded on the teaching of Desargues. Two of the results are important as well as interesting. The first of these is the theorem known now as "Pascal's theorem," that if a hexagon be inscribed in a conic, the points of intersection of the opposite sides will lie in a straight line. The second, which is really due to Desargues, is that if a quadrilateral be inscribed in a conic, and a straight line be drawn cutting the sides taken in order in the points A, B, C, and D and the conic in P and Q, then

$$PA \,.\, PC : PB \,.\, PD = QA \,.\, QC : QB \,.\, QD.$$

Pascal's *Arithmetical triangle* was written in 1653, but not printed till 1665. The triangle is constructed as in the

1	1	1	1	1	......
1	2	3	4	5	......
1	3	6	10	15	......
1	4	10	20	35	......
1	5	15	35	70	......
⋮	⋮	⋮	⋮	⋮	

annexed figure, each horizontal line being formed from the one above it by making every number in it equal to the sum of those above and to the left of it in the row immediately above; e.g. in the 4th line $20 = 1 + 3 + 6 + 10$. Then Pascal's arithmetical triangle (to any required order) is got by drawing a diagonal downwards from right to left as in the figure. These numbers are what are now called *figurate* numbers. Those in the first line are called numbers of the first order; those in the second line, natural numbers or numbers of the second order; those in the third line numbers of the third order, and so on. It is easily shewn that the mth number in the nth row is

$$(m+n-2)!/(m-1)!\,(n-1)!$$

The figures in any diagonal give the coefficients of the expansion of a binomial, e.g. the figures in the fifth diagonal viz. 1, 4, 6, 4, 1, are the coefficients in the expansion $(a+b)^4$. Pascal used the triangle partly for this purpose and partly to find the numbers of combinations of m things taken n at a time, which he stated (correctly) to be

$$(n+1)(n+2)(n+3)\ldots m/(m-n)!$$

Perhaps as a mathematician Pascal is best known in connection with his correspondence with Fermat in 1654 in which he laid down the principles of the *theory of probabilities.* This correspondence arose from a problem proposed by a gamester, the Chevalier de Méré, to Pascal who communicated it to Fermat. The problem was this. Two players of equal skill want to leave the table before finishing their game. Their scores and the number of points which constitute the game being given, in what proportion should they divide the stakes? Pascal and Fermat agreed on the answer, but gave different proofs. The following is a translation of Pascal's solution. That of Fermat is given later.

"The following," says Pascal, "is my method for determining the share of each player, when, for example, two players play a game of three points and each player has staked 32 pistoles.

"Suppose that the first player has gained two points and the second player one point; they have now to play for a point on this condition, that if the first player gains he takes all the money which is at stake, namely 64 pistoles; while if the second player gains each player has two points, so that they are on terms of equality, and if they leave off playing each ought to take 32 pistoles. Thus, if the first player gains then 64 pistoles belong to him, and if he loses then 32 pistoles belong to him. If therefore the players do not wish to play this game, but to separate without playing it, the first player would say to the second 'I am certain of 32 pistoles even if I lose this game, and as for the other 32 pistoles perhaps I shall have them and perhaps you will have them; the chances

are equal. Let us then divide these 32 pistoles equally and give me also the 32 pistoles of which I am certain.' Thus the first player will have 48 pistoles and the second 16 pistoles.

"Next, suppose that the first player has gained two points and the second player none, and that they are about to play for a point; the condition then is that if the first player gains this point he secures the game and takes the 64 pistoles, and if the second player gains this point the players will then be in the situation already examined, in which the first player is entitled to 48 pistoles, and the second to 16 pistoles. Thus if they do not wish to play, the first player would say to the second 'If I gain the point I gain 64 pistoles; if I lose it I am entitled to 48 pistoles. Give me then the 48 pistoles of which I am certain, and divide the other 16 equally, since our chances of gaining the point are equal.' Thus the first player will have 56 pistoles and the second player 8 pistoles.

"Finally, suppose that the first player has gained one point and the second player none. If they proceed to play for a point the condition is that if the first player gains it the players will be in the situation first examined, in which the first player is entitled to 56 pistoles; if the first player loses the point each player has then a point, and each is entitled to 32 pistoles. Thus if they do not wish to play, the first player would say to the second 'Give me the 32 pistoles of which I am certain and divide the remainder of the 56 pistoles equally, that is, divide 24 pistoles equally.' Thus the first player will have the sum of 32 and 12 pistoles, that is 44 pistoles, and consequently the second will have 20 pistoles."

Pascal proceeds next to consider the similar problem when the game is won by whoever first obtains $m + n$ points and one player has m while the other has n points. The answer is obtained by using the arithmetical triangle. The general solution (in which the skill of the players is unequal) is given in any modern text-book on algebra and agrees with Pascal's result, though of course the notation of the latter is different and far less convenient or expressive.

Pascal made a most illegitimate use of the new theory in the seventh chapter of his *Pensées*. He practically puts his argument that as the value of eternal happiness must be infinite, then even if the probability of a religious life ensuring it is very small, still the expectation (which is measured by the product of the two) must be of sufficient magnitude to make it worth while to be religious. I think it was de Morgan who pointed out that the argument, if worth anything, would apply equally to any religion which promised eternal happiness to those who accepted its doctrines. If any conclusion may be drawn from the statement it is the undesirability of applying mathematics to questions of morality of which some of the data are necessarily outside the range of an exact science. It is only right to add that no one had more contempt than Pascal for those who changed their opinions according to the prospect of material benefit, and this isolated passage is at variance with the whole spirit of his writings.

The last mathematical work of Pascal was that on *The cycloid* in 1658. The cycloid is the curve traced out by a point on the circumference of a circular hoop which rolls along a straight line. Galileo in 1630 had been the first to call attention to this curve, and had suggested that the arches of bridges* should be built in the form of it. Four years later Roberval found its area; Descartes thought very little of this solution and defied him to find its tangents; the same challenge was also sent to Fermat, who easily solved the problem. Several questions connected with the curve, and with the surface and volume generated by its revolution about its axis, base, or the tangent at its vertex were then proposed by various mathematicians. These and some analogous questions as well as the positions of the centres of the mass of the solids formed were solved by Pascal in 1658, who issued the results as a challenge to the world. Wallis succeeded in solving all the questions except those connected with the centre

* The only bridge in which I know of this having been done is the one built by Essex in 1766 in the grounds of Trinity College, Cambridge.

of mass. Pascal's own solutions were effected by the method of indivisibles, and correspond exactly with the methods which a modern mathematician would use by the aid of the integral calculus. He obtained by summation what are equivalent to the following integrals

$$\int \sin\phi\, d\phi, \quad \int \sin^2\phi\, d\phi, \quad \int \phi \sin\phi\, d\phi,$$

one limit being either 0 or $\frac{1}{2}\pi$. These researches according to d'Alembert form a connecting link between the geometry of Archimedes and the infinitesimal calculus of Newton.

John Wallis, born at Ashford on Nov. 22, 1616 and died at Oxford on Oct. 28, 1703, was the son of a clergyman, and was educated at Emmanuel College, Cambridge, from which he obtained a fellowship at Queens' College. He subsequently took orders, but on the whole adhered to the puritan party to whom he rendered great assistance by deciphering the royalist despatches. He signed the remonstrance against the execution of Charles I., and thus gave offence to the Independents, but in spite of their opposition he was in the next year, 1649, elected to the Savilian professorship of geometry at Oxford, a chair which he continued to occupy till his death. Besides his mathematical works he wrote on theology and moral philosophy, and he was the first to devise a system for teaching deaf-mutes.

The most important of his works is his *Arithmetica infinitorum* published in 1656. In this treatise the methods of analysis of Descartes and Cavalieri were systematized and greatly extended. It at once became the standard book on the subject; and Fermat, Barrow, Newton, and Leibnitz all constantly refer to it. Wallis commences it by shewing that $x^0, x^{-1}, x^{-2}\ldots$ stand for $1, \frac{1}{x}, \frac{1}{x^2}\ldots$; that $x^{\frac{1}{2}}$ stands for the square root of x, and that $x^{\frac{2}{3}}$ stands for the cube root of x^2 &c. and that thus the law of indices in algebra is quite general.

Leaving the numerous algebraical applications of this discovery he proceeds to find by the method of indivisibles the area enclosed between the curve $y = x^m$, the axis of x, and any ordinate $x = h$; and he shews that this is to the parallelogram on the same base and of the same altitude in the ratio $1 : m + 1$. He seems to assume that the same result would also be true for the curve $y = ax^m$, where a is any constant. In this result m may be any number positive or negative, and he considers in particular the case of the parabola in which $m = 2$ and that of the hyperbola in which $m = -1$. In the latter case his interpretation of the result is incorrect. He then shews that similar results can be written down for any curve of the form $y = \Sigma ax^m$; so that if the ordinate y of a curve can be expanded in powers of the abscissa x its quadrature can be determined. Thus he says that if the equation of a curve is $y = x^0 + x^1 + x^2 + \ldots$ its area will be $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots$. He then goes on to apply this to the quadrature of the curves, $y = (1 - x^2)^0$, $y = (1 - x^2)^1$, $y = (1 - x^2)^2$, $y = (1 - x^2)^3$, &c. taken between the limits $x = 0$ and $x = 1$: and shews that the areas are respectively

$$1, \quad \tfrac{2}{3}, \quad \tfrac{8}{15}, \quad \tfrac{16}{35}, \quad \&c.$$

He next considers curves of the form $y = x^{\frac{1}{m}}$ and shews that the area bounded by the curve, the axis of x, and the ordinate $x = 1$, is to the area of the rectangle on the same base and of the same altitude as $m : m + 1$. This is equivalent to finding the value of $\int_0^1 x^{\frac{1}{m}} dx$. He illustrates this by the parabola in which $m = 2$. He states, but does not prove, the corresponding result for a curve of the form $y = x^{p/q}$.

Wallis shewed great ingenuity in reducing curves to the forms given above, but as he was unacquainted with the binomial theorem he could not effect the quadrature of the circle, whose equation is $y = (1 - x^2)^{\frac{1}{2}}$, since he was unable to expand this in powers of x. He laid down however the principle of interpolation. He argued that as the ordinate of the circle is the geometrical mean between the ordinates of the

curves $y = (1 - x^2)^0$ and $y = (1 - x^2)^1$, so as an approximation its area might be taken as the geometrical mean between 1 and $\frac{2}{3}$. This is equivalent to taking $4\sqrt{\frac{2}{3}}$ or $3 \cdot 26\ldots$ as the value of π. But, he continues, we have in fact a series $1, \frac{2}{3}, \frac{8}{15}, \frac{16}{35}, \ldots$; and thus the term interpolated between 1 and $\frac{2}{3}$ ought to be so chosen as to obey the law of this series. This by an elaborate method leads to a value for the interpolated term which is equivalent to making

$$\pi = 2 \cdot \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \ldots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \ldots}.$$

The subsequent mathematicians of the seventeenth century constantly used interpolation to obtain results which we should attempt to obtain by direct algebraic analysis.

A few years later in 1659 Wallis published a tract in which incidentally he explained how the principles laid down in his *Arithmetica infinitorum* could be used for the rectification of algebraic curves; and in the following year one of his pupils, by name William Neil, applied the rule to rectify the semi-cubical parabola $x^3 = ay^2$. This was the first case in which the length of a curved line was determined by mathematics, and as all attempts to rectify the ellipse and hyperbola had (necessarily) been ineffectual, it had previously been generally supposed that no curves could be rectified. The cycloid was the second curve rectified; this was done by Wren (*Phil. Trans.* 1673). This work of Wallis contains the solution of the problems on the cycloid which had been proposed by Pascal (see p. 255).

In 1665 Wallis published the first systematic treatise on *Analytical conic sections*. I have already mentioned how difficult it is to understand the *Geometry* of Descartes, and to most of his contemporaries, to whom the method was new, it must have been incomprehensible. Wallis made the method intelligible to all mathematicians. This is the earliest book in which these curves are considered and defined as curves of the second degree and not as sections of a cone.

The theory of the collision of bodies was propounded by

the Royal Society in 1668 for the consideration of mathematicians. Wallis, Wren, and Huygens sent correct and similar solutions, all depending on what is now known as the conservation of momentum; but while Wren and Huygens confined their theory to perfectly elastic bodies, Wallis considered also imperfectly elastic bodies.

In 1669 he wrote a work on statics (centres of gravity), and in 1670 one on dynamics: these provide a convenient synopsis of what was then known on the subject.

In 1686 he published an *Algebra*, preceded by a historical account of the development of the subject which contains a great deal of valuable information and in which he seems to have been scrupulously fair in trying to give the credit of different discoveries to their true originators. Among other interesting problems Wallis treats at length (vol. II. p. 472) of the puzzle known as the Chinese rings*, of which Cardan had been the first to give a description; but Wallis' analysis was not equal to the general solution.

This algebra is noteworthy as containing the first systematic use of formulae. A given magnitude is here represented by the numerical ratio which it bears to the unit of the same kind of magnitude: thus when Wallis wants to compare two lengths he regards each as containing so many units of length. This will perhaps be made clearer if I say that the relation between the space described in any time by a particle moving with a

* This puzzle consists of a number of rings hung upon a bar in such a manner that the ring at one end (say A) can be put on or off the bar at pleasure; but any other ring can only be put on or off when the next one to it towards A is on, and all the rest towards A off the bar. The order of the rings cannot be changed. It is easy to show by induction that if there be n rings, it will be necessary, in order to disconnect them from the bar, to put a ring either off or on $\frac{1}{3}(2^{n+1}-1)$ or $\frac{1}{3}(2^{n+1}-2)$ times according as n is odd or even; e.g. if there be sixty rings it will be necessary to put a ring on or off 768614336404564650 times. M. Gros has recently published a most ingenious solution in which the act of taking a ring off or on is represented by the subtraction or addition of unity to a certain number expressed in the binary scale. See *La théorie du baguenodier*, Lyons, 1872.

uniform velocity would be denoted by Wallis by the formula $s=vt$, where s is the number representing the ratio of the space described to the unit of length; while previous writers would have denoted the same relation by stating what is equivalent to the proposition $s_1 : s_2 = v_1t_1 : v_2t_2$; see e.g. Newton's *Principia*, bk. I. sect. i., lemma 10 or 11.

Wallis' mathematical works were collected and published at Oxford in 3 vols. 1697—1699.

While Descartes was laying the foundations of analytical geometry, the same subject was occupying the attention of another and hardly less distinguished Frenchman. This was Fermat. *Pierre de* **Fermat**, who was born near Montauban in 1601 and died at Toulouse on Jan. 12, 1665, was the son of a leather-merchant; he was educated at home; in 1631 he obtained the post of councillor for the local parliament at Toulouse, and he discharged the duties of the office with scrupulous accuracy and fidelity. There, devoting most of his leisure to mathematics, he spent the remainder of his life — a life which but for a somewhat acrimonious dispute with Descartes on the validity of his analysis was unruffled by any event which calls for special notice. The dispute was chiefly due to the obscurity of Descartes, but the tact and courtesy of Fermat brought it to a friendly conclusion. He was a good scholar and amused himself by conjecturally restoring the work of Apollonius on *plane loci*.

Except for a few isolated papers Fermat published nothing in his lifetime, and gave no systematic exposition of his methods. Some of the most striking of his results were found after his death on loose sheets of paper or written in the margins of works he had read and annotated, and are unaccompanied by any proof. It is thus somewhat difficult to estimate the dates or originality of his work. After his death his works and correspondence were published by his nephew at Toulouse in 2 vols. 1670 and 1679. This had long been very scarce, but a summary of it with notes was published by

Brassine at Toulouse in 1853, and a reprint of it was issued at Berlin in 1861. A new edition is now being issued by the French government, which will include several letters on his discoveries and methods in the theory of numbers recently discovered at Leyden by M. Charles Henry and printed in the *Bullettino di bibliografia* for 1879, pp. 477–532 and 735–740. Fermat was constitutionally very modest and retiring, and does not seem to have intended his papers to be published. It is probable that he revised his notes as occasion required, and that his published works represent the final form of his researches, and cannot therefore be dated much earlier than 1660. I shall consider separately (i) his investigations in the theory of numbers; (ii) his use of infinitesimals; and (iii) his method of treating questions of probability.

(i) *The theory of numbers* appears to have been the favorite study of Fermat. He prepared an edition of Diophantus, and the notes and comments thereon contain numerous theorems of considerable elegance: this forms the first of the two volumes of his works. Most of the proofs of Fermat are lost, and it is possible that some of them were not rigorous—an induction by analogy and the intuition of genius sufficing to lead him to correct results. The following include those of his discoveries which are most celebrated.

(*a*) If p is a prime and a is prime to p, then $a^{p-1}-1$ is divisible by p, i.e. $a^{p-1}-1 \equiv 0$ (mod. p). The proof of this (which was first given by Euler) is well known. A more general statement of the theorem is that $a^{\phi(n)}-1 \equiv 0$ (mod. n), where a is prime to n and $\phi(n)$ is the number of integers less than n and prime to it.

(*b*) A prime (greater than 2) can be expressed as the difference of two square integers in one and only one way. Fermat's proof is as follows. Let n be the prime, and suppose it equal to x^2-y^2 i.e. to $(x+y)(x-y)$. Now by hypothesis the only integral factors of n are n and unity, hence $x+y=n$ and $x-y=1$. Solving these equations we get $x=\frac{1}{2}(n+1)$ and

$y=\frac{1}{2}(n-1)$. This theorem has been recently used to find the prime factors of $2^{64}+1$.

(c) To the proposition of Diophantus quoted on p. 102 that the sum of the squares of three integers can never be expressed as the sum of two squares (of which proposition Fermat was the first to give a proof) he added the corollary that it is impossible that any multiple of a prime of the form $(4n-1)$ by a number prime to it, can either be a square or the sum of two squares, integral or fractional. For example 44 is a multiple of 11 (which is of the form $4\times 3-1$) by a number prime to 11, hence it cannot be expressed as the sum of two squares.

(d) A number of the form a^2+b^2 where a is prime to b cannot be divided by a prime of the form $4n-1$.

(e) Every prime of the form $4n+1$ is expressible, and that in one way only, as the sum of two squares. This problem was first resolved by Euler who shewed that a number of the form $2^m(4n+1)$ can always be expressed as the sum of two squares.

(f) If a, b, c are integers such that $a^2+b^2=c^2$, then ab cannot be a square. Lagrange gave a solution of this.

(g) Having given any integer n which is not a square to find a number x such that x^2n+1 may be a square.

(h) There is only one integral solution of the equation $x^2+2=y^3$; and there are only two integral solutions of the equation $x^2+4=y^3$. The required solutions are evidently for the first equation $x=5$, and for the second equation $x=2$ and $x=11$. This question was issued as a challenge to the English mathematicians.

(i) There is no integral solution of the equation $x^n+y^n=z^n$ if n is an integer greater than 2. This last theorem has acquired extraordinary celebrity from the fact that no general demonstration of it has ever been given. Euler in his *De partitione numerorum* proved it when n is equal to 3; and Lagrange in the *Nouv. Mém.* for 1777 gave a proof when n is equal to 4. It appears to be true generally, and Kummer has by means of ideal primes proved it to be so for all except a few special cases. His proof is complicated and difficult, and

it is certain that it is not the same as that discovered by Fermat. The riddle therefore still awaits a solution.

The process adopted by Fermat to prove these results seems to have been one of induction or as he calls it *la méthode de la descente infinie*. It is described in a letter sent by Fermat to Carcavi and now in the university library at Leyden; it is undated, but it would appear from the quotation given below that at the time he wrote it he had only proved the proposition (i) above for the case when $n = 3$. The paper is printed at length in the *Bullettino di bibliografia* for 1879, pp. 737—740; it is too long for me to reproduce textually, but the following extracts will give an idea of Fermat's methods.

Je ne m'en servis au commencement que pour demonstrer les propositions negatives, comme par exemple, qu'il n'y a aucun nombre moindre de l'unité qu'un multiple de 3 qui soit composé d'un quarré et du triple d'un autre quarré. Qu'il n'y a aucun triangle rectangle de nombres dont l'aire soit un nombre quarré. La preuve se fait par ἀπαγωγὴν τὴν εἰς ἀδύνατον en cette manière. S'il y auoit aucun triangle rectangle en nombres entiers, qui eust son aire esgale à un quarré, il y auroit un autre triangle moindre que celuy la qui auroit la mesme proprieté. S'il y en auoit un second moindre que le premier qui eust la mesme proprieté il y en auroit par un pareil raisonnement un troisieme moindre que ce second qui auroit la mesme proprieté et enfin un quatrieme, un cinquieme etc. a l'infini en descendant. Or est il qu'estant donné un nombre il n'y en a point infinis en descendant moindres que celuy la, j'entens parler tousjours des nombres entiers. D'ou on conclud qu'il est donc impossible qu'il y ait aucun triangle rectangle dont l'aire soit quarré. Vide folia post sequens....

Je fus longtemps sans pouvoir appliquer ma methode aux questions affirmatives, parce que le tour et le biais pour y venir est beaucoup plus malaisé que celuy dont je me sers aux negatives. De sorte que lors qu'il me falut demonstrer que tout nombre premier qui surpasse de l'unité un multiple de 4, est composé de deux quarrez je me treuuay en belle peine. Mais enfin une meditation diverses fois reiterée me donna les lumieres qui me manquoient. Et les questions affirmatives passerent par ma methode à l'ayde de quelques nouueaux principes qu'il y fallust joindre par necessité. Ce progres de mon raisonnement en ces questions affirmatiues estoit tel. Si un nombre premier pris à discretion qui surpasse de l'unité un multiple de 4 n'est point composé de deux quarrez il y aura un nombre premier de mesme nature moindre que le donné; et ensuite un troisieme encore moindre, etc. en descendant a l'infini jusques a ce

que nous arrivions au nombre 5, qui est le moindre de tous ceux de cette nature, lequel il s'en suivroit n'estre pas composé de deux quarrez, ce qu'il est pourtant d'où on doit inferer par la deduction à l'impossible que tous ceux de cette nature sont par consequent composez de 2 quarrez. Il y a infinies questions de cette espece.

Mais il y en a quelques autres qui demandent de nouveaux principes pour y appliquer la descente, et la recherche en est quelques fois si mal aisée, qu'on n'y peut venir qu'avec une peine extreme. Telle est la question suivante que Bachet sur Diophante avoüe n'avoir jamais peu demonstrer, sur le sujet de laquelle M.^r Descartes fait dans une de ses lettres la mesme declaration, jusques là qu'il confesse qu'il la juge si difficile, qu'il ne voit point de voye pour la resoudre. Tout nombre est quarré, ou composé de deux, de trois, ou de quatre quarrez. Je l'ay enfin rangée sous ma methode et je demonstre que si un nombre donné n'estoit point de cette nature il y en auroit un moindre qui ne le seroit pas non plus, puis un troisieme moindre que le second &c. à l'infini, d'où l'on infere que tous les nombres sont de cette nature....

J'ay ensuite consideré certaines questions qui bien que negatives ne restent pas de recevoir tres-grande difficulté, la methode pour y pratiquer la descente estant tout a fait diverse des precedentes comme il sera aisé d'esprouver. Telles sont les suivantes. Il n'y a aucun cube divisible en deux cubes. Il n'y a qu'un seul quarré en entiers qui augmenté du binaire fasse un cube, ledit quarré est 25. Il n'y a que deux quarrez en entiers lesquels augmentés de 4 fassent cube, lesdits quarrez sont 4 et 121....

Aprés avoir couru toutes ces questions la pluspart de diverses (*sic*) nature et de differente façon de demonstrer, j'ay passé a l'invention des regles generales pour resoudre les equations simples et doubles de Diophante. On propose par exemple 2 quarr. + 7967 esgaux a un quarré (hoc est $2xx + 7967 \infty$ quadr.) J'ay une regle generale pour resoudre cette equation si elle est possible, ou decouvrir son impossibilité. Et ainsi en tous les cas et en tous nombres tant des quarréz que des unitéz. On propose cette equation double $2x + 3$ et $3x + 5$ esgaux chacun à un quarré. Bachet se glorifie en ses commentaires sur Diophante d'avoir trouvé une regle en deux cas particuliers. Je la donne generale en toute sorte de cas. Et determine par regle si elle est possible ou non....

Voila sommairement le conte de mes recherches sur le sujet des nombres. Je ne l'ay escrit que parce que j'apprehende que le loisir d'estendre et de mettre au long toutes ces demonstrations et ces methodes me manquera. En tout cas cette indication servira aux sçavants pour trouver d'eux mesmes ce que je n'estends point, principalement si M.^r de Carcavi et Frénicle leur font part de quelques demonstrations par la descente que je leur ay envoyees sur le sujet de quelques propositions

negatiues. Et peut estre la posterité me scaura gré de luy avoir fait connoistre que les ancieus n'ont pas tout sceu, et cette relation pourra passer dans l'esprit de ceux qui viendront apres moy pour traditio lampadis ad filios, comme parle le grand Chancelier d'Angleterre, suiuant le sentiment et la deuise duquel j'adjousteray, multi pertransibunt et augebitur scientia.

(ii) Fermat's use of infinitesimals. It would seem from Fermat's correspondence with Descartes as if he had thought out the principles of analytical geometry for himself before reading Descartes' *Discours*, and had realized that from the equation of a curve (or as he calls it, the "specific property") all its properties could be deduced. His extant papers on this subject deal however only with the application of infinitesimals to geometry; it seems probable that these papers are a revision of his original manuscripts (which he destroyed) and were written about 1663, but he was certainly in possession of the general idea of his method for finding maxima and minima as early as 1628 or 1629.

Kepler had already remarked that the values of a function immediately adjacent to and on either side of a maximum (or minimum) value must be equal. Fermat applied this to a few examples. Thus to find the maximum value of $x(a-x)$ he took a consecutive value of x, namely $x-e$ where e is very small, and put $x(a-x)=(x-e)(a-x+e)$. Simplifying and ultimately putting $e=0$ he got $x=\frac{1}{2}a$. This value of x makes the given expression a maximum. The above is the principle of Fermat's method, but his analysis is more involved.

He obtained the subtangent to the ellipse, cycloid, cissoid, conchoid, and quadratrix by making the ordinates of the curve and a straight line the same for two points whose abscissæ were x and $x-e$; but there is nothing to indicate that he was aware that the process was general, and though in the course of his work he used the principle, it is probable that he never separated it, so to speak, from the symbols of the particular problem he was considering. The first definite statement of the method was due to Barrow and was published in 1669 (see p. 269).

Finally Fermat obtained the areas of parabolas and hyperbolas of any order, and determined the centre of mass of a few simple curves and of a paraboloid of revolution. As an example of his method of solving these questions I will quote his solution of the problem to find the area between the parabola $y^n = px^m$, the axis of x, and the line $x=a$. He says that if the ordinates at the points for which x is equal to a, $a(1-e)$, $a(1-e)^2$, ... are drawn then the area will be split into a number of little rectangles whose areas are respectively

$$ae(pa^m)^{\frac{1}{n}}, \quad ae(1-e)\{pa^m(1-e)^m\}^{\frac{1}{n}}, \ \ldots\ldots.$$

The sum of these is $\dfrac{p^{\frac{1}{n}} a^{\frac{m+n}{n}} e}{1-(1-e)^{\frac{m+n}{n}}}$; and by a subsidiary proposition (for of course he was not acquainted with the binomial theorem) he finds the limit of this when e vanishes to be $\frac{n}{m+n} p^{\frac{1}{n}} a^{\frac{m+n}{n}}$. These last theorems were only published after his death; and they were probably not written till after he had read the works of Cavalieri and Wallis.

(iii) Fermat must share with Pascal the honour of having founded *the theory of probabilities.* I have already mentioned (see p. 253) the problem proposed to Pascal, and which he communicated to Fermat, and have there given Pascal's solution. Fermat's solution depends on the theory of combinations and will be sufficiently illustrated by the following example the substance of which is taken from the correspondence with Pascal and is dated Aug. 24, 1654. Suppose, he says, that there are two players, and that the first wants two points to win and the second three points. The game will then certainly be decided in the course of four trials. Take the letters a and b and write down all the combinations that can be formed of four letters. These combinations are the following, 16 in number:

a	a	a	a	a	b	a	a	b	a	a	a	b	b	a	a
a	a	a	b	a	b	a	b	b	a	a	b	b	b	a	b
a	a	b	a	a	b	b	a	b	a	b	a	b	b	b	a
a	a	b	b	a	b	b	b	b	a	b	b	b	b	b	b

Now let A denote the player who wants two points, and B the player who wants three points. Then in those 16 combinations every combination in which a occurs twice or oftener represents a case favorable to A, and every combination in which b occurs three times or oftener represents a case favorable to B. Thus on counting them it will be found that there are 11 cases favorable to A, and 5 cases favorable to B; and as these cases are all equally likely, A's chance of winning the game is to B's chance as 11 is to 5.

The only other problem on this subject which as far as I know attracted the attention of Fermat was also proposed to him by Pascal and was as follows. A person undertakes to throw a six with a die in eight throws; supposing him to have made three throws without success, what portion of the stake should he be allowed to take on condition of giving up his fourth throw? Fermat's reasoning is as follows. The chance of success is $\frac{1}{6}$, so that he should be allowed to take $\frac{1}{6}$ of the stake on condition of giving up his throw. But if we wish to estimate the value of the fourth throw before any throw is made; then the first throw is worth $\frac{1}{6}$ of the stake; the second is worth $\frac{1}{6}$ of what remains, that is $\frac{5}{36}$ of the stake; the third throw is worth $\frac{1}{6}$ of what now remains, that is $\frac{25}{216}$ of the stake; the fourth throw is worth $\frac{1}{6}$ of what now remains, that is $\frac{125}{1296}$ of the stake.

Fermat does not seem to have carried the matter much further, but his correspondence with Pascal shows that he had clear and accurate views on the fundamental principles of the subject: those of Pascal are not altogether correct.

Fermat's reputation is quite unique in the history of science. The problems on numbers which he had proposed long defied all efforts to solve them, and most of them only yielded to the skill of Euler, Lagrange, and Cauchy. One still remains unsolved. This extraordinary achievement has overshadowed his other work, but in fact it is all of the very highest order of excellence, and we can only regret that he thought fit to write so little.

Isaac Barrow was born in London in 1630 and died at Cambridge in 1677. He went to school first at Charterhouse, where he was so troublesome that his father prayed that if it pleased God to take any of his children he could best spare Isaac; and subsequently to Felstead, where it is said he was very industrious. He completed his education at Trinity College, Cambridge; after taking his degree in 1648, and getting a fellowship in 1649, he resided for a few years in college, but in 1655 he was driven out by the persecution of the Independents. He spent the next four years in the East of Europe, and after many adventures, piratical and otherwise, returned to England in 1659. He was ordained the next year, and appointed to the professorship of Greek at Cambridge. In 1662, he was made professor of geometry at Gresham College, and in 1663, was selected as the first occupier of the Lucasian chair at Cambridge. He resigned the latter to his pupil Newton in 1669 whose superior abilities he recognized and frankly acknowledged. For the remainder of his life he devoted himself to the study of divinity. He was appointed master of Trinity College in 1672, and died in 1677. He was noted for his strength, courage, and wit; and was a great favorite of Charles II., but the courtiers could not forgive him for being slovenly in his dress and an inveterate smoker. In appearance he was small in size, lean, and pale.

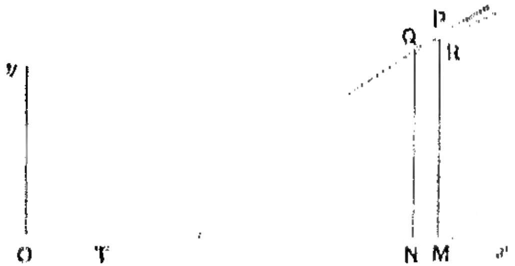
His earliest work was a complete edition of the *Elements* of Euclid in 1660. His lectures, delivered in 1664–6, were published in 1683 under the title *Lectiones mathematicae*; these are mostly on the metaphysical basis for mathematical truths. His lectures for 1667 were published in the same year, and suggest the analysis by which Archimedes was led to his chief results. In 1675 he published an edition with numerous comments of the first four books of the *Conics* of Apollonius, and of the extant works of Archimedes and Theodosius. In 1669 he issued his *Lectiones opticae et geometricae*; this, which is his only important work, was republished with a few minor alterations in 1674. A complete edition of all Barrow's

lectures was edited for Trinity College by W. Whewell, Cambridge, 1860.

It is said in the preface to the *Lectiones opticae et geometricae* that Newton revised and corrected these lectures adding matter of his own, but it seems probable from Newton's remarks in the fluxional controversy that the additions were confined to the parts which dealt with optics.

In the optical lectures many problems connected with the reflexion and refraction of light are treated with great ingenuity. The geometrical focus of a point seen by reflexion or refraction is defined; and it is explained that the image of an object is the locus of the geometrical foci of every point on it. Barrow also worked out a few of the easier properties of thin lenses; and considerably simplified the Cartesian explanation of the rainbow.

The geometrical lectures contain some new ways of determining the areas and tangents of curves. The most celebrated



of these is the method given for the determination of tangents to curves. Fermat had observed that the tangent at a point P on a curve was determined if one other point besides P on it was known; hence if the length of the subtangent MT could be found (thus determining the point T) then the line TP would be the required tangent. Now Barrow remarked that if the abscissa and ordinate at a point Q adjacent to P were drawn he got a small triangle PQR (which he called the differential

triangle, because its sides PR and PQ were the differences of the abscissas and ordinates of P and Q), so that

$$TM : MP = QR : RP.$$

To find $QR : RP$ he supposed that x, y were the coordinates of P, and $x-e$, $y-a$ those of Q (Barrow uses p for x and m for y but I alter these to agree with the modern practice). Using the equation of the curve and neglecting the squares and higher powers of e and a as compared with their first powers he obtained $e : a$. The ratio a/e was subsequently (in accordance with a suggestion made by de Sluze) termed the angular coefficient of the tangent at the point.

Barrow applied this method to the following curves (i) $x^2(x^2+y^2)=r^2y^2$; (ii) $x^3+y^3=r^3$; (iii) $x^3+y^3=rxy$, called *la galande*; (iv) $y=(r-x)\tan \pi x/2r$, the *quadratrix*; and (v) $y=r\tan \pi x/2r$. It will be sufficient here if I take as an illustration the simpler case of the parabola $y^2=px$. Using the notation given above we have for the point P, $y^2=px$; and for the point Q, $(y-a)^2=p(x-e)$. Subtracting we get $2ay-a^2=pe$. But if a is an infinitesimal quantity, a^2 must be infinitely smaller and may therefore be neglected: hence $e : a = 2y : p$. Therefore $TM : y = e : a = 2y : p$. That is $TM = 2y^2/p = 2x$. This is exactly the procedure of the differential calculus, except that we there have a rule by which we can get the ratio $\frac{a}{e}$ or $\frac{dy}{dx}$ directly without the labour of going through a calculation similar to the above for every separate case.

Christian Huygens was born at the Hague on April 14, 1629 and died in the same town on June 8, 1695. He always wrote his name as Hugens, but I follow the usual custom in spelling it as above. It is sometimes written as Huyghens. His life was uneventful and is a mere record of the dates of his various works.

In 1651 he published an essay in which he showed the fallacy in a system of quadratures proposed by Grégoire de

Saint-Vincent (see p. 275) who was well versed in the geometry of the Greeks but had not grasped the essential points in the more modern methods. This essay was followed by tracts on the quadrature of the conics and the approximate rectification of the circle.

In 1654 Huygens' attention was directed to the improvement of the telescope. In conjunction with his brother he devised a new and better way of grinding and polishing lenses. As a result of these improvements he was able during the following two years 1655 and 1656 to resolve numerous astronomical questions; as for example the nature of Saturn's appendage.

His astronomical observations required some exact means of measuring time, and he was thus led to invent the pendulum clock described in his *Horologium*, 1656. The time-pieces previously in use had been balance-clocks.

In the same year, 1656, he wrote a small work on the calculus of probabilities founded on the correspondence of Pascal and Fermat. He spent a couple of years in England about this time. His reputation was now so great that in 1665 Louis XIV. offered him a pension if he would live in Paris, which accordingly then became his place of residence.

In 1668 he sent a paper to the Royal Society of London in answer to a challenge they had issued in which (simultaneously with Wallis and Wren) he proved by experiment that the momentum in a certain direction before the collision of two bodies is equal to the momentum in that direction after the collision. This was one of the points in mechanics on which Descartes had been mistaken.

The most important of Huygens' works was his *Horologium oscillatorium* published at Paris in 1673. The first chapter is devoted to pendulum clocks. The second chapter contains a complete account of the descent of heavy bodies under their own weights in a vacuum, either vertically down or on smooth curves. Amongst other propositions he shews that the cycloid is tautochronous. In the third chapter he

defines evolutes and involutes, proves some of their more elementary properties, and illustrates his methods by finding the evolutes of the cycloid and the parabola. These are the earliest instances in which the envelope of a moving line was determined. In the fourth chapter he solves the problem of the compound pendulum, and shews that the centres of oscillation and suspension are interchangeable. In the fifth and last chapter he discusses again the theory of clocks, points out that if the bob of the pendulum were made by means of cycloidal checks to oscillate in a cycloid the oscillations would be isochronous; and finishes by shewing that the centrifugal force on a body which moves in a circle of radius r with a uniform velocity v varies directly as v^2 and inversely as r.

This was the first attempt to apply dynamics to bodies of finite size and not merely to particles.

In 1674 he designed the watch, the motive power being a spiral spring; and the first watch constructed was made at Paris under his directions, and presented by him to Louis XIV.

The increasing intolerance of the Catholics led to his return to Holland in 1681, and after the revocation of the edict of Nantes he refused to hold any further communication with France. He now devoted himself to the construction of lenses of enormous focal length: of these three of focal lengths 123 ft., 180 ft., and 210 ft. were subsequently given by him to the Royal Society of London in whose possession they still remain. It was about this time that he discovered the achromatic eye-piece (for a telescope) which is known by his name.

In 1689 he came from Holland to England in order to make the acquaintance of Newton whose *Principia* had just been published. But he felt himself too old to change his views and was inclined to reject the Newtonian theory as somewhat occult. In his *Cosmotheorus* published after his death he argues in favour of the vortices of Descartes. This is the least able of his works.

On his return in 1690 Huygens published his treatise on *light* in which the undulatory theory was expounded and ex-

plained. Most of this had been written as early as 1678. The general idea of the theory had been suggested by Hooke in 1664, but he had not investigated its consequences in any detail. This publication falls outside the years considered in this chapter, but it may here be briefly said that according to the wave or undulatory theory space is filled with an extremely thin fluid or gas, and light is caused by a series of waves or vibrations in this fluid which are set in motion by the pulsations of the luminous body. From this hypothesis Huygens deduced the laws of reflexion and refraction, explained the phenomena of double refraction, and gave a construction for the extraordinary ray in biaxal crystals; while he found by experiment the chief phenomena of polarization.

The immense reputation and unrivalled powers of Newton led to the universal disbelief in a theory which he rejected, and to the general adoption of Newton's emission theory (see p. 291). It may however also be noted that Huygens' explanation of some phenomena, such as the colours of thin plates, was inconsistent with the results of experiments; nor was it until Young and Wollaston at the beginning of this century revived the theory and modified some of its details that it was generally accepted.

Besides these works Huygens took part in most of the controversies and challenges which then played so large a part in the mathematical world, and wrote several minor tracts. In one of these he investigated the form and properties of the catenary. In another he stated in general terms the rule for finding maxima and minima of which Fermat had made use, and enunciated the proposition that the subtangent of an algebraical curve $f(x, y) = 0$ was equal to yf_y/f_x, where f_y is the derived function of $f(x, y)$ regarded as a function of y. In some posthumous works issued at Leyden in 1703, he further showed how from the focal lengths of the component lenses the magnifying power of a telescope could be determined; and explained some of the phenomena connected with halos and parhelia.

His works were collected and published in 4 vols.; two at Leyden in 1724 and two at Amsterdam in 1728. His scientific correspondence was published at the Hague in 1833.

I should add that almost all his demonstrations, like those of Newton, are rigidly geometrical, and he would seem to have made no use of the differential or fluxional calculus, though he admitted the validity of the methods used therein. Thus even when first written his works were expressed in an archaic language, and received less attention than their intrinsic merits deserved.

I have now traced the development of mathematics for a period which we may take roughly as dating from 1635 to 1675 under the influence of Descartes, Cavalieri, Pascal, Wallis, Fermat, and Huygens. The life and works of Newton are considered in the next chapter, but it must be remembered that he was the contemporary and friend of Wallis, Huygens, and of some of those immediately hereafter mentioned. These mathematicians seem to me to have been so far superior to their contemporaries, and so much more influential than them, that I may dismiss the remaining mathematicians of this time whom I desire to mention with comparatively slight notice. The following is an alphabetical list of the more remarkable among them: the dates given are those of the birth and death of the mathematician to whose name they are appended. *Brouncker*, 1620—1684: *Courcier*, 1604—1692: *de Beaune*, 1601—1652: *de Laloubère*, 1600—1664: *de Sluze*, 1622—1685: *Dulaurens*, 1597—1657: *Frénicle*, 1605—1670: *Gregory*, 1638—1675: *Hooke*, 1635—1703: *Hudde*, 1633—1704: *Kinckhuysen*, 1630—1679: *Mercator*, 1620—1687: *Ricci*, 1619—1692: *Roberval*, 1602—1675: *Roemer*, 1644—1710: *Saint Vincent*, 1584—1667: *Torricelli*, 1608—1647: *Tschirnhausen*, 1631—1708: *van Schooten*, died in 1661: and *Wren*, 1632—1723. In the following notes I have arranged the above-mentioned mathematicians so that as far as possible their chief contributions shall come in chronological order.

Florimond **de Beaune**, born at Blois in 1601 and died in 1652, wrote a commentary on the obscure and difficult analytical geometry of Descartes. He also discussed the superior and inferior limits to the roots of an equation; this was not published till 1659.

Gilles Personier (de) **Roberval**, born at Roberval in 1602 and died at Paris in 1675, described himself from the place of his birth as de Roberval, a seignorial title to which he had no right. He discussed the nature of the tangents to curves (see p. 243), solved some of the easier questions connected with the cycloid, wrote on mechanics, and on the method of indivisibles. He was a professor in the university of Paris, and in correspondence with nearly all the leading mathematicians of this time. A complete edition of his works was published in 1693.

James **Dodson**, mathematical master at Christ's Hospital, born in London in 1597 and died there in 1657, originated the idea of life-assurance; and calculated the values of annuities for given terms of years circ. 1640.

Frans **van Schooten**, to whom we owe our knowledge of Vieta's works, succeeded his father (who had taught mathematics to Huygens) as professor at Leyden in 1646: he brought out an edition of Descartes' Geometry in 1649; and a collection of mathematical exercises in 1657, in which he suggested the use of coordinates in space of three dimensions: he died in 1661.

Grégoire de **Saint-Vincent**, a Jesuit born at Bruges in 1584 and died at Ghent in 1667, discovered the expansion of $\log(1+x)$ in ascending powers of x. Although a circle-squarer he is worthy of mention for the numerous theorems of interest which he discovered in his search after the impossible, and Montucla ingeniously remarks that "no one ever squared the circle with so much ability or (except for his principal object) with so much success." He wrote two works on the subject, one published in 1647 and the other in 1668, which cover some two or three thousand closely printed pages: the fallacy in the quadrature was pointed out by Huygens. An earlier work entitled *Theoremata Mathematica* published in 1624 con-

tains a very clear account of the method of exhaustions. For further details of Saint-Vincent's life and works, see Quetelet, p. 206.

Evangelista **Torricelli**, born at Faenza in 1608 and died in 1647, wrote on the quadrature of the cycloid and conics; the theory of the barometer; the value of gravity found by observing the motion of two weights connected by a string passing over a fixed pulley; and the theory of projectiles. These were all published in 1644.

Johann **Hudde**, burgomaster of Amsterdam, was born there in 1633 and died in the same town in 1704. He wrote two tracts in 1659: one was on the reduction of equations which have equal roots; in the other he stated what is equivalent to the proposition that if $f(x, y)=0$ is the algebraical equation of a curve, then the subtangent is $-y\frac{\partial f}{\partial y}\Big/\frac{\partial f}{\partial x}$; but being ignorant of the notation of the calculus his enunciation is long and involved.

Bernard **Frénicle** *de Bessy*, born in Paris circ. 1605 and died in 1670, wrote numerous and valuable papers on combinations and on the theory of numbers, also on magic squares. He challenged Huygens to solve the following system of equations in integers, $x^2+y^2=z^2$, $x^2=u^2+v^2$, $x-y=u-v$. This challenge and the correspondence to which it gave rise was only recently discovered. A solution was given by M. Pépin in 1880. *Frénicle's* miscellaneous works, edited by Lahire, were published in vol. 5 of the *Mém. de l'Acad.* 1691.

Antoine de **Laloubère**, a Jesuit, born in Languedoc in 1600 and died at Toulouse in 1664, gave an incorrect solution of Pascal's problems on the cycloid, 1660: he was the first to study the properties of the helix.

Gerard **Kinckhuysen**, born in Holland in 1630 and died in 1679, wrote a text-book on analytical conics in 1660, an algebra in 1661, and formed a collection of geometrical problems solved by analytical geometry. His algebra was edited by Newton in 1669.

Pierre Courcier, a Jesuit, born at Troyes in 1604 and died at Auxerre in 1692, wrote on the curves of intersection of a sphere with a cylinder or cone, also on spherical polygons: the latter was published in 1663.

Michel-Ange Ricci, born in 1619, died at Rome in 1692, was made a cardinal in 1681; wrote a geometry in 1666 in which he solved by Greek geometry those problems on maxima and minima, and on tangents to curves which had been considered by Descartes, Pascal, and Fermat.

Nicholas Mercator was born in Holstein in 1620, but resided most of his life in England: he went to France in 1683, where he designed and constructed the fountains at Versailles, but when they were finished Louis XIV refused to make him the payment agreed on unless he would turn Catholic: he died of vexation and poverty in Paris in 1687. He wrote a treatise on logarithms entitled *Logorithmotechnia* published in 1668, and discovered the series

$$\log(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \ldots;$$

he effected this by writing the equation of the hyperbola in the form

$$y = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$$

to which Wallis' formula (see p. 257) could be applied. The same series had been independently discovered by Saint-Vincent.

William Lord Brouncker, one of the founders of the Royal Society of London, born in 1620 and died on April 5, 1684, was among the most brilliant mathematicians of this time, and was in intimate relations with Wallis, Fermat, and other leading mathematicians. I mentioned on p. 143 his curious reproduction of Brahmagupta's solution of a certain indeterminate equation. Brouncker proved that the area enclosed between the equilateral hyperbola $xy=1$, the axis of x, and the ordinates $x=1$ and $x=2$, is equal to either of the expressions

$$\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \ldots\ldots \text{ or } 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

He also worked out other similar expressions for different areas bounded by the hyperbola and straight lines (*Phil. Trans.* 1672). It is noticeable that he used infinite series to express quantities whose values he could not otherwise determine. In answer to a request of Wallis to attempt the quadrature of the circle he shewed that the area of a circle is to the area of the inscribed square (i.e. $\pi : 2$) in the ratio of

$$\frac{1}{1} + \frac{1^2}{2} + \frac{3^2}{2} + \frac{5^2}{2} + \frac{7^2}{2} + \ldots\ldots : 1.$$

Continued fractions had been introduced by Cataldi in his treatise on finding the square roots of numbers published at Bologna in 1613, but he treated them as common fractions (see p. 210): Brouncker was the first writer who investigated or made any use of their properties.

James Gregory, born at Drumoak near Aberdeen in 1638 and died at Edinburgh in October 1675, was successively professor at St Andrews and Edinburgh. In 1660 he published his *Optica promota* in which the reflecting telescope known by his name is described. In 1667 he issued his *Vera circuli et hyperbolae quadratura* in which he shewed how the areas of the circle and hyperbola could be obtained in the form of infinite convergent series. He added a proof not only that π is incommensurable, but also that the geometrical quadrature of the circle is impossible: Montucla says that the demonstration is correct, de Morgan only remarks that the proof is very abstruse, and expresses no opinion as to its validity. This work by Gregory contains the earliest enunciation of the expansions in series of $\sin x$, $\cos x$, $\sin^{-1} x$, and $\cos^{-1} x$. It was followed in 1668 by the *Geometriae pars* in which Gregory explained how the volumes of solids of revolution could be determined.

Sir Christopher Wren was born at Knoyle in 1632 and died in London in 1723. Wren's reputation as a mathematician has been overshadowed by his skill and fame as an architect. But he was Savilian professor at Oxford from 1661 to 1673,

and for some time president of the Royal Society. Together with Wallis and Huygens he investigated the laws of collision of bodies (*Phil. Trans.* 1669); and he also discovered the two systems of generating lines on the hyperboloid of one sheet (*Phil. Trans.* 1669). Besides these he communicated papers on the resistance of fluids, and the motion of the pendulum. He was a friend of Newton and (like Huygens, Hooke, Halley, and others) had made attempts to shew that the law of force under which the planets moved varied inversely as the square of the distance from the sun.

Wallis, Brouncker, Wren, and Boyle* (the last-named being a chemist and physicist rather than a mathematician) were the leading philosophers who founded the Royal Society of London. The society arose from the self-styled "indivisible college" in London in 1645; most of its members moved to Oxford during the civil wars, where Hooke, who was then an assistant in Boyle's laboratory, joined in their meetings; the society was formally constituted in London in 1660; and was incorporated on July 15, 1662.

Robert **Hooke**, born at Freshwater on July 18, 1635 and died in London on March 3, 1703, was educated at Westminster, and Christ Church, Oxford, and in 1665 became professor of geometry at Gresham College, a post which he occupied till his death. He is still known by the law which he discovered that the tension exerted by a stretched string is (within certain limits) proportional to the extension, or as it is better stated that the stress is proportional to the strain. He invented and discussed the conical pendulum, and was the

* The honourable Robert Boyle, born at Lismore on Jan. 25, 1627, educated at Eton College, and died in London on Dec. 30, 1691, created modern chemistry, and was amongst the earliest of modern experimental physicists. He suggested the freezing and boiling points of water as fixed points for the graduation of thermometers; and discovered in 1662 the law connecting the pressure and density of a gas kept at a constant temperature; his results were confirmed by Mariotte in France in 1676. His life was written by T. Birch, London, 1744.

first to state explicitly that the motions of the heavenly bodies were merely dynamical problems. He was as jealous as he was vain and irritable, and accused both Newton and Huygens of unfairly appropriating his results, but it is probable that he discovered some of their theorems independently. Like Huygens, Wren, and Halley he made efforts to prove that the law of force under which the planets moved about the sun was that of the inverse square. He invented the watch, and had one made in London in 1675; it was finished just three months later than the one made under the directions of Huygens in Paris.

René Francois Walter de Sluze, canon of Liège, born on July 7, 1622 and died on March 19, 1685, introduced the notation of f_x and f_y for the derived functions of $f(x, y)$ with regard respectively to x and y, and wrote numerous tracts especially on spirals, points of inflexion, &c.

Ehrenfried Walter Tschirnhausen, was born at Kislingswalde on April 10, 1631 and died at Dresden on Oct. 11, 1708. In 1682 he worked out the theory of caustics by reflexion, or as they were usually called catacaustics, and showed that they were rectifiable. This was the second case in which the envelope of a moving line was determined (see p. 272). He constructed burning mirrors of great power.

Olof Roemer, born at Aarhuus on Sept. 25, 1644 and died at Copenhagen on Sept. 19, 1710, was the first to measure the velocity of light: this was done in 1675 by means of the eclipses of Jupiter's satellites. He was also the first to introduce micrometers and reading microscopes into an observatory, and it was on his recommendation that astronomical observations of stars were subsequently made in general on the meridian. He also discussed the best form of the teeth in toothed-wheels.

CHAPTER XVI.

THE LIFE AND WORKS OF NEWTON.

SECTION 1. *The life of Newton.*
SECTION 2. *Analysis of Newton's works.*

THE mathematicians considered in the last chapter commenced the creation of those processes which distinguish modern mathematics. The extraordinary abilities of Newton enabled him within a few years to perfect the more elementary of those processes, and to distinctly advance every branch of mathematical science then studied, as well as to create several new subjects. There is hardly a branch of modern mathematics which cannot be traced back to him and of which he did not revolutionize the treatment. Nearly all this work was done between the years 1665 and 1686, though most of it was not printed till some years later. The *Principia* was published in 1687; the rest of his researches were circulated either in manuscript or in the transactions of the Royal Society, but the bulk of them were ultimately issued in book form between the years 1704 and 1709.

In pure geometry Newton did not establish any new methods, but no modern writer has ever shown the same power in using those of classical geometry, and he solved many problems in it which had previously baffled all attempts. In algebra and the theory of equations he introduced the system of literal indices, established the binomial theorem (in 1669), and created no inconsiderable part of the theory of equations. One rule

which he enunciated in this subject remained till a few years ago as an unsolved riddle which had overtaxed the resources of all succeeding mathematicians. Newton always, by choice, avoided using trigonometry in his analysis, and I do not think he ever published anything on that subject. In analytical geometry he introduced the modern classification of curves into algebraical and transcendental; and established many of the fundamental properties of asymptotes, multiple points, and isolated loops. He illustrated these by an exhaustive discussion of cubic curves.

The fluxional or infinitesimal calculus was invented by Newton in or before the year 1666, and circulated in manuscript amongst his friends in and after the year 1669, though no account of the method was printed till 1693.

Newton further was the first to place dynamics on a thoroughly satisfactory basis, and from dynamics he deduced the theory of statics: this was in the introduction to the *Principia* published in 1687. The theory of attractions, the application of the principles of mechanics to the solar system, the creation of physical astronomy, and the establishment of the law of universal gravitation are wholly due to Newton and were first published in the same work. The particular questions connected with the motion of the earth and moon were worked out as fully as was then possible. The theory of hydrodynamics was created by him in the second book of the *Principia*, and he also added considerably to the theory of hydrostatics which may be said to have been first discussed by Pascal. The theory of the propagation of waves and in particular the application to determine the velocity of sound is due to Newton and was published in 1687.

In geometrical optics, he explained amongst other things the decomposition of light and the theory of the rainbow; he invented the reflecting telescope known by his name, and the sextant. In physical optics he was the author of the emission theory of light.

This list of the subjects he discussed and the theorems he

invented is by no means exhaustive, but I preface this chapter with it in order to emphasize the fact that Newton more than any one else is the creator of modern mathematics, and that his investigations placed the subject in a completely new position.

The life of Newton.

Isaac Newton* was born in Lincolnshire near Grantham on Dec. 25, 1642 (O. S.), and died at Kensington, London, on March 20, 1727. He was educated at Trinity College, Cambridge, and lived there from 1661 till 1697 during which years he produced the bulk of his work in mathematics; he was then appointed to a valuable government office, and moved to London where he resided till his death. I shall follow the course I have usually adopted, and first give a short account of his life, and then a brief account of his works.

His father, who had died shortly before Newton was born, was a yeoman farmer, and it was intended that Newton should carry on the paternal farm. He was sent to school at Grantham. At first he was very lazy, but a fight with a boy above him in the school in which he was victorious led him to determine to try to be equally successful in learning. He soon became head of the school, and his learning and mechanical proficiency excited some attention: as one instance of his ingenuity I may mention that he constructed a clock worked by water which kept very fair time. In 1656 he returned home to learn the business of a farmer under the guidance of an old family servant. Newton however spent most of his time solving problems, making experiments, or devising mechanical models; his mother noticing this sensibly resolved to find some more congenial occupation for him, and sent him back to school again. Here his uncle met him and being

* The chief authorities for Newton's life and works are discussed in *The Memoirs of Newton*, by D. Brewster, 2 vols, Edinburgh (2nd ed.), 1860; and *The History of the Inductive Sciences*, by W. Whewell, Cambridge, 1837.

development of the subject. As however the leading facts are generally known, and the works published during this time are accessible to any student, I may deal more concisely with the lives and writings of modern mathematicians than with those of their predecessors.

Roughly speaking we may say that five distinct stages in the history of this period can be discerned.

First of all there is the invention of analytical geometry by Descartes in 1637; and almost at the same time the introduction of the method of indivisibles, by the use of which areas, volumes, and the positions of centres of mass can be determined by summation in a manner analogous to that effected now-a-days by the aid of the integral calculus. The method of indivisibles was soon superseded by the integral calculus. Analytical geometry however maintains its position as part of the necessary training of every mathematician, and is incomparably more potent than the geometry of the ancients for all purposes of research. The latter is still no doubt an admirable intellectual training, and it frequently affords an elegant demonstration of some proposition the truth of which is already known, but it requires a special procedure for every problem attacked. The former on the other hand lays down a few simple rules by which any property can be at once proved or disproved.

In the *second* place we have the invention of the fluxional or differential calculus about 1666 (and possibly an independent invention of it in 1674). Wherever a quantity changes according to some continuous law (and most things in nature do so change) the differential calculus enables us to measure its rate of increase or decrease; and from its rate of increase or decrease the integral calculus enables us to find the original quantity. Formerly every separate function of x such as $(1+x)^n$, $\log(1+x)$, $\sin x$, $\tan^{-1}x$, &c. could only be expanded in ascending powers of x by means of such special procedure as was suitable for that particular problem; but by the aid of the calculus the expansion of any function of x in ascending

himself a Trinity man recommended that he should go up to Cambridge.

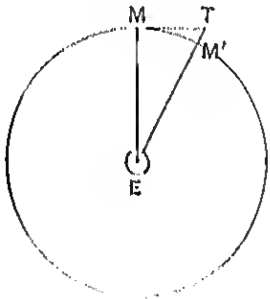
In 1661 he accordingly entered as a subsizar at Trinity College, where for the first time he found himself among surroundings which were likely to develop his powers. He seems however to have had little interest for general society or for any pursuits save science and mathematics, and he complained to his friends that he found the other undergraduates disorderly. He got his scholarship in 1663, and took his B.A. degree in 1664.

We know that he had read Sanderson's Logic before coming up to Cambridge. His mathematical reading there was as follows. He bought a book on astrology at Stourbridge Fair in October 1661, but could not understand it on account of the geometry and trigonometry; he therefore got a Euclid, and was surprised to find how obvious the propositions seemed. He thereupon read Oughtred's Clavis and Descartes' Geometry, the latter of which he managed to master by himself though with some difficulty. The interest he felt in the subject led him to take up mathematics rather than chemistry as a serious study. His subsequent mathematical reading as an undergraduate was founded on Kepler's Optics, the works of Vieta, Schooten's Miscellanies, Descartes' Geometry, Wallis' Arithmetica infinitorum, and Barrow's lectures. At a later time on reading Euclid more carefully he formed a very high opinion of it as an instrument of education, and he often expressed his regret that he had not applied himself to geometry before proceeding to algebraic analysis. He made some optical experiments and observations on lunar halos while an undergraduate.

There is a manuscript of his written in the year following his degree, and dated May 28, 1665, which is the earliest documentary proof of his discovery of fluxions; the general idea of the calculus was therefore probably invented in the year 1665. It was about the same time that he discovered the binomial theorem (see p. 293).

On account of the plague the college was sent down in the summer of 1665, and for the next year and a half Newton lived at home. This period was crowded with brilliant discoveries. He worked out the fluxional calculus tolerably completely: thus in a manuscript dated Nov. 13 of the same year he uses fluxions to find the tangent and the radius of curvature at any point on a curve, and in October 1666 he applies them to several problems in the theory of equations. Newton communicated the results to his friends and pupils from and after 1669, but they were not published in print till 1693. From this the celebrated controversy with Leibnitz arose which is mentioned later (see pp. 328—333).

He also at this time, that is in 1666, thought out the elements of his theory of gravitation. Leaving out details and taking round numbers only, his reasoning is usually said to have been as follows. He knew that if a stone were allowed to fall near the surface of the earth the attraction of the earth (that is the weight of the stone) caused it to move through



16 feet in one second. Now he knew the distance of the moon, and therefore the length of its path. He also knew the time the moon took to go once round it, namely a month. Hence he could easily find its velocity at any point such as M. He could therefore find the distance MT through which it would

move in the next second if it were not pulled by the earth's attraction. At the end of that second it was however at M', and therefore the earth must have pulled it through the distance TM' in one second (assuming the direction of the earth's pull to be constant). Now he and several physicists of the time had conjectured from Kepler's third law that the attraction of the earth on a body would be found to decrease as the body was removed further away from the earth in a proportion inversely as the square of the distance from the centre of the earth*; but up to this time no one knew how to test whether the guess was true or not. If however this was the actual law then TM' should be to 16 feet in a proportion which was inversely as the square of the distance of the moon from the centre of the earth to the radius of the earth. When however Newton made the calculation he found that TM' was about one-eighth less than it ought to have been on this hypothesis. It seemed therefore as if this was not the true law of attraction, and Newton put the investigation on one side. When in 1684 he became acquainted with more accurate data than those he had used in this work, TM' was found to have exactly the value which was required by the hypothesis and the verification was complete.

The above is the usually received account of his investigations on gravity in 1666, but Prof. Adams (than whom no greater authority could be obtained) says that there can be no doubt that the numerical verification was fairly complete in 1666, and that Newton's difficulty was of a totally different kind. If I understand him aright, he says that Newton was then firmly convinced of the principle of universal gravitation, that is, that every particle of matter attracts every other par-

* The reasoning was as follows. If v be the velocity of a planet, r the radius of its orbit taken as a circle, and T its periodic time, $v=2\pi r/T$. But if f is the acceleration to the centre of the circle,

$$f=\frac{v^2}{r}=\frac{4\pi^2 r}{T^2}.$$

Now by Kepler's third law T^2 varies as r^3; hence f varies inversely as r^2.

ticle. He probably suspected that the attraction varied as the product of their masses and inversely as the square of the distance between them, but he had not been able to determine what the attraction of a spherical mass on any external point would be, and it is at any rate certain, as we know from his letters to Halley, that he did not then suppose the attraction of the earth to act as if it were concentrated into a single particle at its centre nor did he believe that this was in the least likely to be the case. He must therefore have regarded the relation between gravity and the moon's motion as showing that the moon was kept in its orbit by the attraction of the earth, and not as giving the exact law of attraction, though it made it probable that at a considerable distance the latter varied approximately as the inverse square.

It was also while staying at home at this time that he devised some instruments for grinding lenses to particular forms other than spherical, he perhaps decomposed light, and he certainly devoted considerable time to astrology and alchemy. I may add that he never abandoned the idea of transmuting base metals into gold, though I do not think he attempted to effect it by resolving gold into its elements: whether the latter experiment is possible has yet to be proved.

On his return to Cambridge in 1667 Newton got a fellowship, and took his M.A. degree in 1668. He took pupils, and it is probable that his *Universal Arithmetic* which was a manuscript on algebra, theory of equations, and miscellaneous problems (see p. 331) was of this date. It was printed by Whiston against his wishes, but with his consent, in 1707. His note books show that his attention was now mostly occupied with chemistry and optics, though there are a good many problems in pure and analytical geometry scattered amongst them. In 1668 he constructed a reflecting telescope in order to evade the difficulties of chromatism caused by the use of lenses.

In the early part of 1669, or perhaps in 1668 he revised Barrow's lectures for him (see p. 269). The end of lecture XIV. is known to have been written by Newton, but how much

of the rest is due to his suggestions cannot now be determined.

As soon as this was finished he was asked by Barrow and Collins to revise and add notes to a translation of Kinckhuysen's *Algebra* (see p. 276): he consented to do this on condition that his name should not appear in the matter. He was elected Lucasian professor in 1669, and the preparation of his lectures occupied most of his time in that year and the beginning of 1670; but he finished the algebra before the close of 1670. He also at Collins' request solved some problems on harmonic series and on annuities which had previously baffled investigation, though with that morbid dislike to publicity which coloured all his life, he only gave permission that his results should be published "so it be" as he says "without my name to it: for I see not what there is desirable in public esteem, were I able to acquire and maintain it: it would perhaps increase my acquaintance, the thing which I chiefly study to decline."

In 1670 he also wrote his analysis by infinite series, the object of which was to express the ordinate of a curve in an infinite algebraical series every term of which could be integrated by Wallis' rule (see p. 257). This was given to Barrow, and he by letters dated June 20, July 31, and Aug. 21 communicated it (with Newton's leave) to Collins. In the early part of 1671 Newton began a systematic exposition of his analysis by series. It was never finished, but translations of the fragment were published in 1736 and 1745.

In these two years he had thus revised and edited Barrow's *Lectures*, edited and added to Kinckhuysen's *Algebra*, and by using infinite series greatly extended the power of the method of quadratures given by Wallis. These however were only the fruits of his leisure; most of his time during these years being given up to optical researches.

In October 1669 Barrow had resigned the Lucasian chair in favour of Newton. Newton chose optics for the subject of his lectures and researches, and before the end of the year he had worked out the details of his discovery of the decomposition

of a ray of white light into rays of different colours, which was effected he tells us by means of a prism bought at Stourbridge Fair. The manner in which he was led to these investigations seems to have been as follows. It was well known that the images formed by lenses were indistinct. Descartes, James Gregory, and others hoped to correct this by grinding the surfaces to aplanatic forms, and this was what Newton had himself tried to do in 1666. He now however determined to see whether there might not be some other cause for this indistinctness besides the fact that the rays from any point of the object were not brought accurately to a focus at a single point. The whole investigation is a model of rigorous scientific reasoning, and the conclusion was that white light was not homogeneous but consisted of rays of different refrangibilities. The complete explanation of the theory of the rainbow followed from this discovery.

By a curious chapter of accidents Newton failed to correct the chromatic aberration of two colours by means of a couple of prisms. He therefore abandoned the hope of making a refracting telescope which should be achromatic, and instead designed a reflecting telescope, probably on the model of the small one previously alluded to. The form he invented is that still known by his name. The Royal Society heard of this in 1670 or 1671 and asked that it might be sent to London; it was accordingly presented to the society in 1671. Newton does not seem to have ever resumed the practical construction of telescopes: though in 1672 he invented a reflecting microscope.

In 1671 Newton began to prepare an edition of the lectures on optics, twenty in number, which he had as Lucasian professor delivered in the years 1669, 1670, and 1671. These were not printed till 1729, after his death, when a manuscript copy of them which he had given to David Gregory, the Savilian professor at Oxford, was published (see p. 336). The larger part of his new results in optics up to this date were however incorporated in the papers communicated to the Royal Society

at its request in January and February 1672, and published in its transactions. His deductions from these experiments were attacked with considerable vehemence by Pardies in France, Linus and Lucas at Liege, Hooke in England, and Huygens in Paris; but his opponents were finally silenced and convinced. The correspondence which this entailed on Newton occupied nearly all his leisure in the years 1672 to 1675 and prevented his doing any original work. Writing on Dec. 9, 1675, he says "I was so persecuted with discussions arising out of my theory of light, that I blamed my own imprudence for parting with so substantial a blessing as my quiet to run after a shadow." Again on Nov. 18, 1676, he observes "I have made myself a slave to philosophy; but if I get rid of Mr Linus's business, I will resolutely bid adieu to it eternally, excepting what I do for my private satisfaction, or leave to come out after me; for I see a man must either resolve to put out nothing new, or to become a slave to defend it."

He was also harassed at this time about his future, as his fellowship lapsed in 1675. The Crown however on April 27, 1675, gave him a special patent to continue to hold it as long as he was professor; and he was about the same time excused payment of his subscriptions to the Royal Society, of which since 1671 he had been a fellow.

Not only was his time during those years taken up with the controversy on the validity of his own conclusions and experiments, but Hooke involved him in a correspondence about the cause of light. Newton rejected the wave or emission theory proposed by Hooke, but (without committing himself to any belief in it) suggested in 1673 certain improvements in the theory.

This correspondence seems to have suggested to Newton the enquiry as to how light was really produced, and in the spring of 1675, as soon as his financial affairs were arranged, he set himself to examine this problem. By the close of the year he had worked out the corpuscular or emission theory, and the results were communicated to the Royal Society on Dec. 9 and

Dec. 16 of that year. I should add that some of Newton's experiments were made between 1672 and 1675, and that Hooke had previously observed the colours of thin plates. The only other paper he wrote on optics was in 1687 in which he elaborates the theory of fits of easy reflexion and transmission, the inflexion of light (bk. II. part 1), and the colours of thick plates (bk. II. part 4). The three papers together contain the whole of his theory of light, and comprise the bulk of his treatise on optics published in 1704, to which the references given immediately above refer.

The object of these two papers of 1675 was to establish the emission theory. Only three ways have been suggested in which light can be produced mechanically. Either the eye may be supposed to send out something which, so to speak, feels the object (as Euclid had supposed, see p. 56); or the object perceived may send out something which hits or affects the eye (as Newton supposed in his corpuscular or emission theory); or there may be some medium between the eye and the object, and the object may cause some change in the form or nature of this intervening medium and thus affect the eye (as Hooke and Huygens supposed in the wave or undulatory theory). It will be enough here to say that on either of the two latter theories all the obvious phenomena of geometrical optics such as reflexion, refraction, &c. can be accounted for. Within the present century crucial experiments have been devised which give different results according as one or the other theory is adopted; all these experiments agree with the results of the undulatory theory and differ from the results of the Newtonian theory: the latter is therefore untenable, but whether the former represents the whole truth and nothing but the truth is still an open question. Until however the theory of interference was worked out by Young the hypothesis of Huygens failed to account for all the facts and was open to more objections than that of Newton. Although Newton did not believe that the wave theory was the true explanation, he subsequently elaborated the fundamental prin-

ciples of it in the eighth section of the second book of tl *Principia.*

I may so far depart from the chronological order as t say that besides some minor papers and experiments on optic Newton invented the sextant in 1700. He sent the descrip tion of the latter to Halley, who attached so little importanc to it that he did not communicate it to the Royal Societ Newton's MS. was found amongst Halley's papers on the deat of the latter in 1742. The instrument was independentl discovered in 1731 by a certain John Hadley.

Two letters written by Newton in the year 1676 a sufficiently interesting to justify an allusion to them. Leibni who had been in London in 1673 had just begun to stud mathematics seriously, and had communicated some resul to the Royal Society which he had supposed to be new, b which it was pointed out to him had been previously prove by Mouton. This led to a correspondence with Oldenbur the secretary of the Society. In 1674 Leibnitz wrote sayi that he possessed "general analytical methods depending o infinite series." Oldenburg in reply told him that Newto and Gregory had used such series in their work. In answe to a request for information Newton wrote on June 13, 167 giving a very brief account of his method but adding tl expansions of a binomial (i.e. the binomial theorem) and o $\sin^{-1} x$; from the latter of which he deduced that of $\sin x$. H also added an expression for the rectification of an elliptic a in an infinite series.

Leibnitz wrote on Aug. 27 asking for fuller details, and o Oct. 24 Newton replied in a long but very interesting pap in which he gives an account of the way in which he had be led to some of his results.

He begins by saying that altogether he had used thr methods for expansion in series. His first was arrived a from the study of the method of interpolation by which Wall had found expressions for the area of the circle and hyperbol Thus, by considering the series of expressions

$$(1-x^2)^{\frac{0}{2}},\ (1-x^2)^{\frac{2}{2}},\ (1-x^2)^{\frac{4}{2}},\ \&c.$$

he deduced by interpolations the law which connects the successive coefficients in the expansions of

$$(1-x^2)^{\frac{1}{2}},\ (1-x^2)^{\frac{3}{2}},\ \&c.$$

He then by analogy obtained the expression for the general term in the expansion of a binomial, i.e. the binomial theorem. He says that he proceeded to test this by forming the square of the expansion of $(1-x^2)^{\frac{1}{2}}$ which reduced to $1-x^2$; and he proceeded in a similar way with other expansions. He next tested the theorem in the case of $(1-x^2)^{\frac{1}{2}}$ by extracting the square root of $1-x^2$ *more arithmetico*. He also used the series to determine the areas of the circle and hyperbola in infinite series and found that they were the same as the results he had arrived at by other means.

Having established this result he then discarded the method of interpolation, and employed his binomial theorem as the most direct method of obtaining the areas and arcs of curves. Newton styled this his second method and it is the basis of his work on analysis by infinite series. He states that he had discovered it before the plague in 1665—66, and goes on to say that on being then obliged to leave Cambridge he had ceased to pursue these ideas as he suspected that Nicholas Mercator had employed some of them in his *Logarithmotechnia*; and he supposed that the remainder would have been found out before he himself was of sufficiently ripe age to publish his discoveries. He had almost forgotten that he had ever used his first method, until in turning over his papers to write to Leibnitz, he had come across the notes he had formerly made on the subject. There appears to be some confusion in this statement as the *Logarithmotechnia* was not published till 1668; but it seems clear that the discovery was made in 1666, though for some reason it was not made known to his friends till 1669.

Newton then proceeds to state that he had also a third

method; of which (he says) he had about 1669 sent an account to Barrow and Collins, illustrated by applications to areas, rectification, cubature, &c. This was the method of fluxions; but Newton gave no detailed description of it in this letter, probably because he thought that Leibnitz could, if he wished, obtain from Collins the explanation of it alluded to above. Newton added an anagram which described the method but which is unintelligible to any one to whom the key is not given. He gives however some illustrations of its use. The first is on the quadrature of the curves represented by

$$y = ax^m (b + cx^n)^p,$$

which he says can be effected as a sum of $\frac{m+1}{n}$ terms if $\frac{m+1}{n}$ be a positive integer, and which he thinks cannot otherwise be effected except by an infinite series. [This is not so, the integration is possible if $p + (m+1)/n$ be an integer.] He also gives a long list of other forms which are immediately integrable of which the chief are

$$\frac{x^{mn-1}}{a + bx^n + cx^{2n}}, \quad \frac{x^{(m+\frac{1}{2})n-1}}{a + bx^n + cx^{2n}},$$

$$x^{mn-1} (a + bx^n + cx^{2n})^{\pm\frac{1}{2}},$$

$$x^{mn-1} (a + bx^n)^{\pm\frac{1}{2}} (a + dx^n)^{-1},$$

and $$x^{(m-1)n-1} (a + bx^n)^{\frac{1}{2}} (a + dx^n)^{-\frac{1}{2}};$$

where m is a positive integer and n is any number whatever.

At the end of his letter Newton alludes to the solution of the "inverse problem of tangents," a subject on which Leibnitz had asked for information. He gives formulae for reversing any series, but says that besides these formulae he has two methods for solving such questions which for the present he will not describe except by an anagram which being read is as follows, "Una methodus consistit in extractione fluentis quantitatis ex aequatione simul involvente fluxionem ejus. Altera tantum in assumptione seriei pro quantitate qualibet incognita ex qua caetera commode derivari possunt, et in

collatione terminorum homologorum æquationis resultantis, ad ernendos terminos assumptæ seriei."

He adds in this letter that he is worried by the questions he is asked and the controversies raised about every new matter which he publishes, and he regrets that he has allowed his repose to be interrupted by running after shadows; and he implies that for the future he will publish nothing. As a matter of fact he did refuse to allow any account of his method of fluxions to be published till the year 1693.

Leibnitz did not reply to this letter till June 21, 1677. In his answer he explains his method of drawing tangents to curves, which he says proceeds "not by fluxions of lines but by the differences of numbers"; and he introduces his notation of *dx* and *dy* for the infinitesimal differences between the coordinates of two consecutive points on a curve. He also gives a solution of the problem to find a curve whose subtangent is constant, which shows that he could integrate.

I do not know what were Newton's occupations during the next eight years, 1676—1684. He was partly engaged in chemical experiments and partly in geological speculations; and I believe, though I speak with some hesitation, that his experiments in electricity and magnetism*, and on the law of cooling are of this date. A large part of the geometry and the pure mathematics which were subsequently incorporated in the first book of the *Principia* should probably be also referred to this time: and perhaps some parts of the essay on cubic curves.

In the latter part of 1679, in consequence of a letter from Hooke relating to projectiles, Newton was led to consider what would be the form of the curve which a body projected from a point and acted upon by a central attraction varying inversely as the square of the distance would describe. He then established the theorem relating to the equal description of areas, and discovered a general method of determining the law of

* He thought that small magnets attracted one another with a force which varied inversely as the cube of the distance between them.

a central force in order that any given orbit might be described under its action. This he applied to the ellipse, and proved that if this curve were described by a particle under the action of a force directed to the focus then this force must vary inversely as the square of the distance from the focus. Conversely he showed that if the force varied as the inverse square of the distance from a point, the orbit would be a conic having that point as focus. Hooke believed that it was an inherent property of a celestial body to attract all matter (whether inside or outside it) to its centre; and he conjectured that the law was always that of the inverse square. Newton stated that it was impossible that that could be the law inside the earth, but it is quite probable that Hooke's letter suggested to him the possibility that for external points that might give exactly the direction and magnitude of the resulting attraction. Newton was however at this time fully occupied with other subjects, so after applying his method to the ellipse and answering Hooke's question, he laid these calculations aside, and gave no thought to the subject again for five years.

In August 1684 Newton received a visit from Halley who drew his attention again to the motion of the moon. Hooke, Huygens, Halley, and Wren had all conjectured that the force of the attraction of the sun or earth on an external particle varied inversely as the square of the distance; and the investigations of Hooke were particularly ingenious. These writers seem to have independently shown that if Kepler's conclusions were rigorously true, as to which they were not quite certain, the law of attraction must be that of the inverse square, but they could not deduce from the law the orbits of the planets. When Halley visited Cambridge in August 1684 he explained that their investigations were stopped by their inability to solve this problem, and asked Newton if he could find out what the orbit of a planet would be if the law of attraction were that of the inverse square. Newton immediately replied that it was an ellipse, and promised to send or write out afresh his old demonstration of it. This was sent in November 1684;

and either it or a fuller exposition of it was registered in December 1684 under the title *De motu*.

It is known that it was in 1684, and no doubt it was immediately after Halley's visit, that Newton reviewed his early attempt to see whether the moon's motion was in accordance with the law of the inverse square. Using some observations of Picard on the dimensions of the earth (which had been communicated to the Royal Society in 1672) he found that at the end of one minute the deflexion of the moon from the line in which it had been moving (i.e. the tangent to its path) was about 16 feet. Now the space described by a falling body in the first second of its motion was also 16 feet. Hence (using the formula $s = \frac{1}{2}ft^2$) the force of the earth's attraction at the distance of the moon was to the force at the surface of the earth in the ratio $1 : (60)^2$; that is inversely as the squares of their respective distances from the centre of the earth. This may only have been a repetition of his rough verification of 1666. At any rate Newton made it again, and thus was enabled to show that if the distances of the members of the solar system were so great that they might for the purpose of their mutual attraction be regarded as points then their motions were in accordance with the law of gravitation.

The elements of these discoveries were put together in the tract called *De motu*, which contains the substance of sections ii. and iii. of the first book of the *Principia*, and was read by Newton for his lectures in the Michaelmas term 1684.

Newton however had not in 1684 determined the attraction of a spherical body on any external point, nor had he calculated the details of the planetary motions even if the members of the solar system could be regarded as points. The first problem was solved in 1685, probably either in January or February. "No sooner," to quote from Dr Glaisher's address on the bicentenary of the publication of the *Principia*, "had Newton proved this superb theorem—and we know from his own words that he had no expectation of so beautiful a result till it emerged from his mathematical investigation—than all the mechanism of the

universe at once lay spread before him. When he discovered the theorems that form the first three sections of book I., when he gave them in his lectures of 1684, he was unaware that the sun and earth exerted their attractions as if they were but points. How different must these propositions have seemed to Newton's eyes when he realized that these results, which he had believed to be only approximately true when applied to the solar system, were really exact! Hitherto they had been true only in so far as he could regard the sun as a point compared to the distance of the planets, or the earth as a point compared to the distance of the moon — a distance amounting to only about sixty times the earth's radius — but now they were mathematically true, excepting only for the slight deviation from a perfectly spherical form of the sun, earth and planets. We can imagine the effect of this sudden transition from approximation to exactitude in stimulating Newton's mind to still greater efforts. It was now in his power to apply mathematical analysis with absolute precision to the actual problems of astronomy." After three or four weeks' rest he devoted himself to the explanation of the detailed phenomena of the solar system and in the almost incredibly short space of time from March 1686 to the end of March 1687 he completed the whole of the *Principia*. Of the three fundamental principles there applied we may say that the idea that every particle attracts every other particle in the universe was formed at least as early as 1666; the law of equable description of areas, its consequences, and the fact that if the law of attraction were that of the inverse square the orbit of a particle about a centre of force would be a conic were proved in 1679; and lastly the discovery that a sphere, whose density at any point depends only on the distance from the centre, attracts an external point as if the whole mass were collected at its centre was made in 1685. It was this last discovery that enabled him to apply the first two principles to the phenomena of bodies of finite size.

The first book of the *Principia* was finished on April 28,

1686. This book is given up to the consideration of the motion of particles or bodies in free space either in known orbits, or under the action of known forces, or under their mutual attraction. In it Newton generalizes the law of attraction into a statement that every particle of matter in the universe attracts every other particle with a force which varies directly as the product of their masses and inversely as the square of the distance between them; and he thence deduces the law of attraction for spherical shells of constant density.

In another three months, that is by the summer of 1686, he had finished the second book of the *Principia*. This book treats of motion in a resisting medium, and of hydrostatics and hydrodynamics, with special applications to waves, tides, and acoustics. He concludes it by showing that the Cartesian theory of vortices was inconsistent both with the known facts and with the laws of motion.

The next nine or ten months were devoted to the third book. For this he probably had no materials ready. In it the theorems obtained in the first book are applied to the chief phenomena of the solar system: the masses and distances of the planets and (whenever sufficient data existed) of their satellites are determined. In particular the motion of the moon, the various inequalities therein, and the theory of the tides are worked out in great detail. He also investigates the theory of comets, shows that they belong to the solar system, explains how from three observations the orbit can be determined, and illustrates his results by considering certain special comets.

The third book of the *Principia* as we have it was but little more than a sketch of what Newton had proposed to himself to accomplish. The original programme of the work that he had set before himself is among the Portsmouth papers, and it is evident that he continued to carry it out for some years after the publication of the *Principia*. It is possible that his investigations were interrupted by his serious illness in 1693 and not resumed, but in any case they were dis-

continued on his leaving Cambridge in 1696. Most of these unpublished researches seem to have been destroyed, but some fragments have been preserved among his papers, which Professor Adams has fortunately been able to decipher. Of these perhaps the most interesting are those in which he has carried his approximations by means of fluxions beyond the point at which he was able to translate them into geometry. His efforts to do so are amongst the Portsmouth papers and will I hope be shortly made public.

The printing of the work was very slow and it was not finally published till the summer of 1687. The whole cost was borne by Halley* who also corrected the proofs and even put his own researches on one side to press the printing forward.

The conciseness, absence of illustrations and synthetical character of the book as first issued seriously restricted the numbers of those who were able to appreciate its value; and though nearly all competent critics admitted the validity and value of the work a considerable time elapsed before it affected the current beliefs of educated men. I should be inclined to say (but on this point opinions differ widely) that within ten years of its publication it was generally accepted in Britain as giving a correct account of the laws of the universe, it was similarly accepted within about twenty years on the continent, except in France where patriotism was urged in defence of the Cartesian theory until Voltaire in 1738 took up the advocacy of the Newtonian theory.

The manuscript of the *Principia* was finished by 1686.

* *Edmund Halley*, born in London in 1656 and died at Greenwich in 1742, was educated at St Paul's School, London, and Queen's College, Oxford, succeeded Wallis in 1703 as Savilian professor, and subsequently in 1720 was appointed astronomer royal in succession to Flamsteed. Most of his works are on astronomy and magnetism. He conjecturally restored the eighth and lost book of the Conics of Apollonius, and in 1710 brought out a magnificent edition of the whole work. He also edited the works of Serenus (issued in 1710), of Menelaus (published in 1758), and some of the minor works of Apollonius. He was in his turn succeeded at Greenwich by Bradley, the most distinguished astronomer of the age, and the discoverer of the cause of astronomical aberration.

Newton devoted the remainder of that year to his paper on optics (see p. 291) in which he completed the emission theory. This paper was read before the Royal Society in 1687: the greater part of it is given up to the subject of diffraction.

In 1687 James II. having tried to force the university to admit as a master of arts a Roman Catholic priest who refused to take the oaths of supremacy and allegiance, Newton took a prominent part in resisting the illegal interference of the king, and was one of the deputation sent to London to protect the rights of the university. The active part taken by Newton in this affair led to his being in 1689 elected member for the university. This parliament only lasted thirteen months, and on its dissolution he gave up his seat. He was subsequently returned in 1701, but he never took any prominent part in politics.

On his coming back to Cambridge in 1690 he resumed his mathematical studies and correspondence. The two letters to Wallis in which he explains his method of fluxions and fluents were written at this time, Aug. 27 and Sept. 17, 1692: they were published in 1693. It was at this time also that he sent a copy of his lectures on optics to David Gregory who on his recommendation had just been appointed Savilian professor at Oxford.

Towards the close of 1692 and throughout the two following years Newton had a long illness, suffering from insomnia and general nervous irritability. It was at one time said that he was going out of his mind, but his correspondence shews no sign of this and the rumour seems to have been the invention of those who were jealous of his fame. He however never regained his elasticity of mind, and though after his recovery he shewed the same power in solving any question propounded to him, he ceased thenceforward to do original work on his own initiative, and it was difficult to stir him to activity. This mental sluggishness is I think very noticeable in the correspondence with Cotes from 1709 to 1713. Although Cotes was editing the *Principia* for Newton he received but little assistance, and his requests for information of

what Newton had meant or how a theorem should be proved were frequently postponed or not answered.

In 1694 Newton began to collect data connected with the irregularities of the moon's motion with the view of revising the part of the *Principia* which dealt with that subject. To render the observations more accurate he forwarded to Flamsteed* a table of corrections for refraction which he had previously made. This was not published till 1721 when Halley communicated it to the Royal Society. The original calculations of Newton and the papers connected with it are in the Portsmouth collection at Cambridge, and shew that Newton obtained it by finding the path of a ray by means of quadratures in a manner equivalent to the solution of a differential equation. As an illustration of Newton's genius I may mention that even as late as 1754 Euler failed to solve the same problem: Laplace in 1782 gave a solution of it, and his results agree substantially with those of Newton.

I do not suppose that Newton would in any case have produced much more original work after his illness; but his appointment in 1695 as warden, and his promotion in 1699 to the mastership of the mint at a salary of £1500 a year, brought his scientific investigations to an end. His knowledge of chemistry and mechanics here proved useful, and he continued to fill the office efficiently till his death in 1727.

The remaining events in his life may be summed up very shortly. In 1701 he resigned the Lucasian chair; in 1703 he was elected president of the Royal Society; in 1704 he published his *Optics* with two appendices. Of these one was on curves of the third degree, in which they are classified and their chief properties investigated. The second was on the

* *John Flamsteed*, born at Derby in 1646 and died at Greenwich in 1719, was one of the most distinguished astronomers of this age. Besides much valuable work in astronomy he invented the system (published in 1680) of drawing maps by projecting the surface of the sphere on an enveloping cone, which can then be unwrapped. He was succeeded as astronomer royal by Halley.

quadrature of curves by expressing the ordinate in terms of the abscissa (if necessary using an infinite series) and contains an account of his method of fluxions. In 1705 he was knighted. From this time onwards he devoted much of his leisure to theology, and wrote at great length on prophecies and predictions which had always been subjects of interest to him.

His *Universal Arithmetic* was published by Whiston in 1707; but Newton had nothing to do with preparing it for the press.

The dispute with Leibnitz as to whether he had derived the ideas of the differential calculus from Newton or invented it independently originated about 1709, and excited great interest especially from the years 1709 to 1716. I allude briefly to this a few pages later.

In 1709 Newton was persuaded to allow Cotes to prepare the long-talked-of second edition of the *Principia*. The first edition had been out of print by 1690; but though Newton had collected some materials for a second and enlarged edition, he could not at first obtain the requisite data from Flamsteed the astronomer royal, and subsequently he was unable or unwilling to find the time for the necessary revision. The correspondence between Newton and Cotes on the various alterations made in this edition is preserved in the library of Trinity College, Cambridge, and is extremely interesting: it was edited by Edleston for the college in 1850. The second edition of the *Principia* was issued in March 1713, and was sold out within a few months, but a pirated edition published at Amsterdam supplied the demand. In 1723 Newton entrusted the editing of a third edition to Henry Pemberton* and this was published in 1726.

In 1725 Newton's health began to fail, and on March 20, 1727, he died of stone. His body was carried to the Jerusalem Chamber, and on the 28th of March was buried with great state in Westminster Abbey.

* *Henry Pemberton*, born in London in 1694 and died at Oxford on March 9, 1771, was professor of physic at Gresham College. He wrote on several points connected with Newton's discoveries, but was best known to his contemporaries for his lectures and text-book on chemistry.

In appearance Newton was short, and towards the close of his life rather stout, but well set, with a square lower jaw, a very broad forehead, rather sharp features, and brown eyes. His hair turned grey before he was thirty, and remained thick and white as silver till his death.

As to his manners. He dressed slovenly, was rather languid, and was generally so absorbed in his own thoughts as to be anything but a lively companion. Many anecdotes of his extreme absence of mind when engaged in any investigation have been preserved. Thus once when riding home from Grantham he dismounted to lead his horse up a steep hill, when he turned at the top to remount he found that he had the bridle in his hand, while his horse had slipped it and gone away. Again on the few occasions when he sacrificed his time to entertain his friends, if he left them to get more wine or for any similar reason, he would as often as not be found after the lapse of some time working out a problem, oblivious alike of his expectant guests and of his errand. He took no exercise, indulged in no amusements, and worked incessantly, often spending 18 or 19 hours out of the 24 in writing.

In character he was perfectly straightforward and honest, but in his controversies with Leibnitz, Hooke, and others though scrupulously just he was not generous. He modestly attributed his discoveries largely to the admirable work done by his predecessors; in answer to a correspondent he explained that if he had seen farther than other men, it was only because he had stood on the shoulders of giants. He was morbidly sensitive to being involved in any discussions. I believe that with the exception of his two papers on optics in 1675, every one of his works was only published under pressure from his friends and against his own wishes. There are several instances of his communicating papers and results on condition that his name should not be published. During the early half of his life he was parsimonious, if not stingy, and he was never liberal in money matters.

In intellect he has never been surpassed and probably never

been equalled. Of this his extant works are the only proper test. Perhaps the most wonderful single illustration of his powers was the composition in seven months of the first two books of the *Principia*. Another example which may strike many people is his solution of the problem of Pappus to find the locus of a point such that the rectangle under its distances from two given straight lines shall be in a given ratio to the rectangle under its distances from two other given straight lines. Nearly all the great geometricians from the time of Apollonius had tried the problem by geometry and had failed, and it was in his efforts to solve it that Descartes was led to the invention of analytical geometry; but what had proved insuperable to all his predecessors seems to have presented little difficulty to Newton who gave an elegant demonstration that the locus was a conic. Geometry, said Lagrange when recommending the study of analysis to his pupils, is a strong bow, but it is one which only a Newton can fully utilize.

To these illustrations of his ability I may add the two following examples.

In 1697 John Bernoulli challenged the world (i) to determine the brachistochrone, and (ii) to find a curve such that if any line drawn from a fixed point O cut it in P and Q then $OP^n + OQ^n$ would be constant. Leibnitz solved the first of these questions after rather more than six months' efforts, and then suggested they should be sent as a challenge to Newton and others. Newton received the problems on Jan. 29, 1697, and gave the complete solutions of both the next day; at the same time generalizing the second question. The answers were sent anonymously through Montague, but Bernoulli recognized the hand of Newton "even as the lion is known by his paw."

An almost exactly similar case occurred in 1716 when Newton was asked to find the orthogonal trajectory of a family of curves. In five hours Newton solved the problem in the form in which it was propounded to him and laid down the principles for finding trajectories.

It is almost impossible to describe the effect of Newton's writings without being suspected of gross exaggeration. But if the state of mathematical knowledge in 1669 or at the death of Pascal or Fermat be compared with what was known in 1687 it will be seen how immense was the advance. In fact we may say that it took mathematicians half a century or more before they were able to master and assimilate the work which Newton had produced in those twenty years.

The influence of Newton is impressed on all the subjects of modern mathematics, and his name is familiar to every mathematical student; but his books are written in a language which is repulsive to most modern readers and the majority of critics are content to take him at second hand. I believe however that without exception those who have consulted his original works and they number the greatest names of subsequent times rank his mathematical achievements as the most wonderful that any one has ever produced.

It will be enough to quote the remarks of two or three of those who were subsequently concerned with the subject-matter of the *Principia*. Lagrange on reading the *Principia* said he felt dazed at such an illustration of what man's intellect might be capable. In describing the effect of his own writings and those of Laplace it was a favourite remark of his that Newton was not only the greatest genius that had ever existed but he was also the most fortunate, for as there is but one universe, it can happen but to one man in the world's history to be the interpreter of its laws. He added that he hoped that the solutions of those problems which were beyond the reach of Newton's time and genius, but which had yielded to the analysis of the subsequent century, should be translated into the language of the *Principia* "in order," to use his own words, "to give to the greatest production of the human mind the perfection of which it is susceptible."

Laplace, who is in general very sparing of his praise, makes of Newton the one exception, and the words in which he enumerates the causes which "will always assure

to the *Principia* a pre-eminence above all the other productions of the human intellect" have often been quoted. Not less remarkable is the homage rendered by Gauss. For all other great mathematicians or philosophers, he used the epithets magnus, or clarus or clarissimus. For Newton alone he kept the prefix summus.

If I add one more quotation it shall be from Biot, who though only a mathematician of the second rank had made a special study of Newton's works. Yet he although writing with the object of minimizing Newton's investigations on the method of fluxions, almost in spite of himself sums up his remarks by saying, "comme géomètre et comme expérimentateur Newton est sans égal; par la réunion de ces deux genres de génies à leur plus haut degré, il est sans exemple."

A complete collection of Newton's works was published by Horsley at London in 1779, but it has long been out of print. Some of his correspondence has been published, but the great mass of his letters and mathematical manuscripts which are now extant, forming the Portsmouth collection, still remain unedited and unpublished. Fortunately their owner in 1872 placed them in the hands of the University of Cambridge, and in the same year Dr Luard with Profs. Stokes, Adams, and Liveing were commissioned to inspect them. Their report can hardly fail to throw fuller light on the record of Newton's work. It is already known that Newton's note books show that he had worked out by means of fluxions and fluents his approximations in the lunar theory to a higher order of approximation than that shown in the *Principia* but was unable to translate his reasoning into the language of geometry. His unsuccessful efforts to do so are among his papers. Even more interesting than this is his own solution of the famous problem of the shape of the solid of least resistance. The construction is given correctly in book II. prop. 25, but the calculus of variations seems to be required to determine it in the first instance, and it has always been a mystery how Newton obtained the result. Professor

Adams has found a draft of a letter to David Gregory in which, in reply to a request for information upon this point, Newton gives two demonstrations.

Analysis of Newton's works.

In order to avoid breaking the continuity of my remarks on Newton's work I contented myself with stating generally the subject-matter of his papers on various subjects. I add here a very brief statement of the contents of the different books he produced, taking them in their order of publication. These are the *Principia*, published in 1687; the *Optics* (with appendices on *cubic curves*, the *quadrature of curves*, and the *method of fluxions*), published in 1704; the *Universal Arithmetic*, published in 1707; the *Lectiones opticae* published in 1729; the *Methodus differentialis* published in 1736; and the *Analytical geometry* also published in 1736.

Before describing the subject-matter of the *Principia* it will be convenient to make a few remarks on the form in which it was presented and which seriously hindered the immediate adoption of the Newtonian philosophy. Not only are the proofs throughout the work geometrical, but no clue is given as to the method by which Newton arrived at them, and there are no illustrations or explanations. The difficulties inherent in such demonstrations are greatly increased by the extreme conciseness of Newton's language, and the omission of numerous steps in the argument which to readers of ordinary ability are by no means obvious. When such brilliant geometricians as Clairaut and Lagrange (who had the aid of the admirable Jesuit commentaries of 1739 and 1742 in which most of the proofs were amplified and illustrated) assert that to follow the reasoning requires concentrated and continuous effort it may be imagined how difficult the work must have seemed to Newton's contemporaries.

The reason why it was presented in a geometrical form appears to have been that the fluxional calculus was unknown

to most of Newton's contemporaries, and had he used it to demonstrate results which were in themselves opposed to the prevalent philosophy of the time the controversy would have first turned on the validity of the methods used. Newton therefore cast the whole reasoning into a geometrical shape which, if somewhat longer, can at any rate be made intelligible to all mathematical students and of which the methods are above suspicion. So closely did he follow the lines of Greek geometry that he constantly used graphical methods and represented all the magnitudes considered (be they forces, velocities, or times) in the Euclidean way by straight lines, e.g. book I., section i., lemma 10, and not by a certain number of units. The latter and modern method had been introduced by Wallis, and must have been familiar to Newton. The effect of his confining himself rigorously to classical geometry and elementary algebra, and his refusal to make any use even of analytical geometry and of trigonometry, is that the *Principia* is written in a language which is archaic (even if not unfamiliar) to us.

The adoption of geometrical methods in the *Principia* for purposes of demonstration does not indicate a preference on Newton's part for geometry over analysis as an instrument of research, for there is no doubt that Newton used the fluxional calculus in the first instance in finding some of the theorems especially those towards the end of book I. and in book II.; and in fact one of the most important uses of that calculus is stated in book II., lemma 2. But it is only just to remark that at the time of its publication and for nearly a century afterwards the differential and fluxional calculus were not fully developed and did not possess the same superiority over the Newtonian method which they do now; and it is a matter for astonishment that when Newton did employ the calculus he was able to use it to so good an effect. This translation of numerous theorems of great complexity into the language of the geometry of Archimedes and Apollonius is I suppose among the most wonderful intellectual feats ever performed.

The *Principia* was published in 1687. After an introduction on dynamics, it is divided into three books, the first of which deals with motion in free space, the second with motion in a resisting medium, and the third with applications to the solar system. It concludes with a general scholium.

The whole is preceded by a preface in which Newton says that his object is to apply mathematics to the phenomena of nature. Among these phenomena motion is one of the most important. Now motion is the effect of force, and though he does not know what is the nature or origin of force, still many of its effects can be measured; and it is these that form the subject-matter of the work.

The work begins therefore naturally with an introduction on dynamics or the science of motion. This commences with eight definitions of various terms such as mass, momentum, &c. Newton then lays down three laws of motion which are incapable of exact proof, but are confirmed partly by direct experiments, partly by the agreement with observation of the deductions from them. From these he deduces six fundamental principles of mechanics, and adds an appendix on the motion of falling bodies, projectiles, oscillations, impact, and the mutual attractions of two bodies. The most important deduction is that of the parallelogram of velocities, accelerations, and forces. The following is a brief account of the Newtonian method of treating dynamics.

The first law asserts that every body will continue in its state of rest or of uniform motion in a straight line except so far as it is compelled to change it by some external force. Hence the definition of force as any cause which alters or tends to alter the state of rest or motion of a body. From this law also is derived the method of comparing different times, for if no force acts on a body it will move uniformly, that is, will pass over equal spaces in equal times: now the earth is a rotating body, and approximately no external force hinders the rotation; hence equal times are those in which the earth turns through equal angles. Again the law asserts that a body is inert or pas-

sive, and does not itself tend to alter its motion; if therefore a body does not move with uniform velocity some force must have acted having a component in the direction of motion; this component is said to overcome the inertia of the body, and by the second law it is measured by the change of momentum produced per unit of time. Lastly if a body does not move in a straight line some force must have acted which has a component perpendicular to the direction of motion; this component is said to overcome the centrifugal force of the body, and it is shown later that it is measured by twice the product of the kinetic energy and the curvature: i.e. by mv^2/ρ.

The second law asserts that change of momentum (per unit of time) is proportional to the impressed force, and takes place in the direction of the force. Hence if at the time t a body of mass m is moving with a velocity v under the action of a force F, then $\frac{d}{dt}(mv) \propto F$. It is usual to choose the units so that $\frac{d}{dt}(mv) = F$. The same fact is sometimes expressed by saying that the total change of momentum produced is equal to the impulse which produces it and is in the same direction; that is $[mv] = \int F dt$, where both sides of the equation are taken between corresponding limits. This law therefore enables us to measure forces. It also enables us to compare masses; for example if two bodies of masses m_1 and m_2 be at rest and if equal forces be applied to them for equal times then the total momentum produced must be the same in each case; if the velocity at the end of the time be v_1 in the mass m_1, and v_2 in the mass m_2, then $m_1v_1 = m_2v_2$, and therefore $m_1 : m_2 = v_2 : v_1$. This would however be a troublesome experiment to make and it is unnecessary, for Newton showed by pendulum experiments (book III., prop. 6) that at any given place the weight of a body was proportional to its mass, and therefore the ratio of two masses is the same as the ratio of their weights. From this law Newton de-

duced the parallelogram of velocities and the parallelogram of forces.

The first and second laws* give all that is required for solving any question on the motion of a particle under the action of given forces.

Newton's third law supplies the principle required for the solution of problems in which two or more particles influence one another. It asserts that the action of one body on another body is equal in magnitude but opposite in direction to the reaction of the second body on the first. Newton gives two ways of interpreting this law. First there is the obvious meaning that action and reaction are equal and opposite. Looked at as parts of the same phenomenon, either the action or reaction is what is now called a stress; a force is therefore one aspect of a stress. A single force is however unknown to us, for whenever a force is caused another equal and opposite one is also brought into existence, though it may act upon a different body, and it may in any particular problem be unnecessary for us to consider it. The second interpretation given by Newton is that in machines the rate at which an agent does work (that is its action) is equal to the rate at

* The following additional definitions are not given explicitly in Newton, but as I shall have to use the terms I add them here. A force is said to do work when it overcomes resistance, i.e. when the body on which it acts moves in the direction of the force: the work done is measured by the product of the average force overcome and the distance through which it is overcome. Energy is power of doing work, and experience shews that it is of two kinds, namely, potential and kinetic. Potential energy is the power of doing work which a body has in consequence of its position with reference to other bodies, or of its configuration; e.g. a coiled spring. It is measured by the work which must be done from some standard position to put the body in that position. Kinetic energy is the power of doing work which a body has in consequence of its motion; e.g. a cannon ball in motion. It is measured by the work which can be got out of its motion before it is reduced to some standard condition (usually relative rest) i.e. if the velocity be only translational by $\frac{1}{2}mv^2$, where m is its mass, and v its velocity of translation.

which work is done against it (that is its reaction) provided that we include in the reaction the rate at which kinetic energy is being produced. As a particular case of the law the "internal forces of inertia" (i.e. the forces which resist acceleration) must be equal and opposite to the actions by which the acceleration is produced. This was first explicitly stated by d'Alembert in 1743 (see p. 353) and is known as his principle.

If this second interpretation had been extended to include work done by or against molecular forces, which of course Newton did not intend, it would have been equivalent to the statement that the work done by an agent on a system is equivalent to the increase of kinetic energy *plus* the increase of potential energy; which is the principle of the conservation of energy. The evidence for the indestructibility of energy is one of the great achievements of modern physics, just as the evidence for the indestructibility of matter is one of the great achievements of modern chemistry. As far as we can tell every thing* in the physical world can be classified either as energy or as matter: within the limits of experience both matter and energy are indestructible, but while the former is inert or passive the latter is only known to us in connection with matter and in the act of changing from one form to another. In Newton's time it was believed that both could be destroyed.

The first book of the *Principia* is on the motion of bodies in free space, and is divided into fourteen sections.

The *first section* consists of eleven preliminary lemmas treated by the method of prime and ultimate ratios, and not by that of indivisibles.

The *second section* commences by showing that if a body (such as a planet) revolve in an orbit subject to a force tending to a fixed point (such as the sun), the areas swept out by radii drawn from the body to the point are in one plane and are

* It is perhaps worth noting that time is a sequence, velocity is a change of position, &c.; but energy is a thing, and like other things is bought and sold.

proportional to the times of describing them: and conversely if the areas be proportional to the times the force acting on the body must be directed to the point. Newton then shows how if the orbit is known and the centre of force is given the law of force can be determined; and he finds the law for several curves.

In the *third section* he applies these propositions to a body which describes a conic section about a focus, and proves that the force must vary inversely as the square of the distance, and that Kepler's third law would necessarily be true of such a system. Conversely he proves that if a body were projected in any way and subject to a force which varied according to this law then it must move in a conic section having the centre of the force in the focus. He concludes (prop. 17, cors. 3 and 4) with a pregnant suggestion as to how the effects of disturbing forces should be calculated: this was first done by the brilliant investigations of Laplace and Lagrange; and Laplace says (*Méc. cél.* book XV., chap. I.) that Lagrange's paper in the Berlin memoirs for 1786 on which the modern treatment of the subject is founded was suggested by these remarks of Newton.

The *fourth* and *fifth sections* are devoted to the geometry of conic sections, especially to the construction of conics which satisfy five conditions. The geometry is throughout extremely ingenious but very concise. In section four one of the conditions is that the focus is given; this includes the problem of finding the path of a comet from three observations which Newton says he found the most difficult problem of any which he had to solve: curiously enough he gave a second solution of this problem in book III. prop. 41 which he recommended as more simple but which is inapplicable in practice.

The *sixth section* is devoted to determining what at any given time is the velocity and what is the position of a body which is describing a given conic about a centre of attraction in a focus: together with various converse problems. To effect this Newton had to find the area of a sector of a conic. This is easily done for the parabola. He then proceeds to "en-

deavour to demonstrate" that exact quadrature of any closed oval curve having no infinite branches (such as the ellipse) is impossible. This proof is not correct as it stands, and Newton seems himself to have felt some doubt about inserting it though he believed the result to be true*. An exact quadrature being impossible he proceeds to give three ways, two arithmetical and one geometrical, of approximating to the sectorial area of an ellipse as closely as is desired.

The *seventh section* is given up to the discussion of motion in a straight line under a force which varies inversely as the square of the distance, and its comparison with motion in a conic under the same force. He concludes by giving a general solution for all the problems considered in this section for any law of force. He here determines geometrically what is equivalent to finding the integral of $x(ax - x^2)^{-\frac{1}{2}}$.

The *eighth section* contains general solutions of the various problems hitherto considered for any orbit and any central force whatever. The reasoning is ingenious and correct but extremely complex. In proposition 40 he states that the kinetic energy acquired by a body in moving from one point to another point is equal to the total work done by the force between those two points.

In the *ninth section* he discusses the case where the orbit is in motion in its own plane round the centre of force, and treats in detail of the motion of the apse-line, and the forces by which a given motion would be produced. Newton applied this reasoning to the case of the moon, but the resulting motion of the apses only came out about one half of the actual amount. The approximation was in fact not carried to a sufficiently high order. Newton was aware of the discrepancy, and as he explained the similar difficulty in the case of the node it had long been suspected (see e.g. Godfray's *Lunar Theory*, 2nd edition, § 68) that the scholium in the first

* Dr Routh (in his *Analytical view of the Principia*, p. 75) has pointed out that it is not true for ovals of the form $y^{2m} = (2n)^{2m} x^{2m(2n-1)} (a^{2n} - x^{2n})$, where m and n are positive integers.

edition to book III, prop. 35 meant that he had found the explanation. Nowhere in the *Principia* does he however give any hint as to how this was effected, and the true explanation of a difference which had long formed an obstacle to the universal acceptance of the Newtonian system was first given by Clairaut in 1752.

The Portsmouth papers, now in the possession of the university of Cambridge, contain Newton's original work, and shew that he had obtained the true value by carrying the approximation to a sufficiently high order. It also seems clear from these papers that Newton gave the corollary as a mere illustration of the motion of the apses in orbits which are nearly circular and did not mean it to apply to the moon, but by an inadvertence he added in the second and third editions a reference to it as an authority for a result connected with the moon which would naturally deceive any reader. Newton left most of the revision of the second edition to Cotes and it is probable that the mistake is due to a blunder of the editor. Other lunar and planetary irregularities are also discussed in this proposition, but the extreme conciseness of Newton misled all the early commentators, and even Laplace in his earlier work of the *Système du monde* published in 1796 speaks of Newton as having only roughly sketched out this part of the subject, leaving it to be completed when the calculus should be further perfected: but in the last volume of his *Mécanique céleste* published in 1825 he says that on more careful reading he has no hesitation in regarding it as among the most profound parts of the work.

The *tenth section* is devoted to the consideration of the motion of bodies along given surfaces, but not in planes passing through the centre of force; with special reference to the vibration of pendulums, and the determination of the accelerating effect of gravity. In connection with the latter problem Newton investigates the chief geometrical properties of cycloids, epicycloids, and hypocycloids: this is very ingenious and the geometry is simple.

In the *eleventh section* are considered the problems connected with motion in orbits where the centre of force is disturbed, or where the moving body is disturbed by other forces. Until the calculus of variations was invented by Lagrange in 1755 it was impossible to do more than sketch out the principles on which the problem should be solved, and Laplace in his *Mécanique céleste* in 1825 was the first to work out most of the questions in any detail, but he and Lagrange considered this section to be on the whole the most remarkable part of the *Principia*, though many of the solutions are only outlined. Newton commences by considering the disturbance produced by the mutual action of two bodies revolving round one another. He then proceeds to consider the problem of three or more bodies which mutually attract one another. He first solves the question completely if the force of attraction varies directly as the distance. He next takes the case of three bodies moving under their mutual attractions as in nature. This problem has not yet been solved generally, but in Newton's day it was beyond any analysis of which he had the command: he contrived however to work out roughly the chief effects of the disturbing action of the sun on the motion of the moon (prop. 66). To this proposition he appended twenty-two corollaries in which he applied it clearly, but with extreme conciseness, to determine the motion in longitude, in latitude, the annual equation, the motion of the apse line, and of the nodes, the evection, the change of inclination, the precession of the equinoxes, and the theory of the tides. The greater part of the third book consists of the numerical application of these principles to the case of the moon and the earth. Lastly Newton showed how from the motion of the nodes the interior constitution of the body could be roughly determined: this proposition was singled out by Lagrange as the most striking illustration of the genius of Newton.

Up to this point Newton had generally treated the bodies with which he dealt as if they were particles. He now proceeds in *section twelve* to consider the attractions of spherical masses

which are either of uniform density, or whose density at any point is a function of the distance of the point from the centre of the sphere. These are worked out for any law of attraction.

In *section thirteen* he gives some general theorems on the theory of attractions and some propositions dealing with the attractions of solids of revolution, but these problems are almost insoluble without the aid of the infinitesimal calculus, and the Newtonian account of them is incomplete.

The *fourteenth section* contains a statement of some of his theories and experiments in physical optics; and a solution by geometry of some difficult problems in geometrical optics, particularly on the form of aplanatic refracting surfaces of revolution.

The second book of the *Principia* is concerned with hydromechanics, and especially with motion in a resisting medium. It shows the same skill and a genius which is almost intuitive as the first book, but it is by no means so carefully finished; and though it provided the basis on which the subsequent work of Daniel Bernoulli, Clairaut, d'Alembert, Euler, and Laplace was erected it is not of the same unapproached excellence as the first book. No other treatise of the same epoch-making character as the first part of the *Principia* has ever been produced: the second part may rank among the half dozen most influential scientific books yet written, but it is not like the first book absolutely unique.

This book is divided into nine sections, and I confine myself to enumerating the subjects therein treated. The motion of bodies in a medium where the resistance varies directly as the velocity is considered in the *first section*. The motion where the resistance varies as the square of the velocity is discussed in the *second section*. The motion where the resistance can be expressed as the sum of two terms, one of which varies as the velocity and the other as the square of the velocity, is dealt with in the *third section*.

The *fourth section* is devoted to spiral motion in a resisting medium. The *fifth* to the motion of pendulums in a resisting

medium. The *sixth* to the theory of hydrostatics. The *seventh* to hydrodynamics, and especially to the motion of projectiles in air. The *eighth* to the theory of waves, including the principles from which the chief effects of the wave hypothesis in light are calculated, and to acoustics.

In the *ninth section* Newton discusses the Cartesian theory of vortices (see p. 246). He begins by shewing that if there were no internal friction the motion would be impossible. He must therefore assume some law of friction, and as a working hypothesis he supposes that "the resistance arising from want of lubricity in the parts of a fluid is, *cæteris paribus*, proportional to the velocity with which the parts of the fluid are separated from each other." This hypothesis, as he himself remarks, is probably not altogether correct, but he thinks that it will give a general idea of the motion. He then proves that on this hypothesis the motion would be unstable. He must therefore suppose that some constraining force prevents this catastrophe, and he then shows that in that case Kepler's third law could not be true. Lastly he shews by independent reasoning that the hypothesis must lead to results which are inconsistent with Kepler's other two laws, and that both the vortices and the motion of the planets would necessarily be unstable. Great efforts were made in France by John Bernoulli, Huygens, Perrault, Villemot, Mollieres, Gamaches and others to modify the Cartesian hypothesis so as to avoid these conclusions, but they could never explain one phenomenon without introducing fresh difficulties. It may be taken that by 1750 the Cartesian theory was finally abandoned.

The third book is headed *On the system of the world* and is chiefly concerned with the application of the results of the first book to the solar system. It is introduced by the "rules of philosophizing," namely (i) we may only assume as the possible causes of phenomena such causes as if admitted would explain them and are also *veræ causæ*; a *vera causa* being one which is capable of detection and such that its connection with the phenomenon can be ultimately shown by independent

evidence; (ii) effects of a similar kind must have similar causes; (iii) whatever properties of bodies are found by experience to be invariable should be assumed to be so in places where direct experiments cannot be made. The book is divided into five sections which are respectively on the causes of the system of the world, on the lunar errors, on the tides, on the precession of the equinoxes, and on comets.

Newton commences by illustrating the universality of the law of gravitation, and sketches out the principles which lead him to think that the solar system is necessarily stable; he next determines the mass of the moon, the masses of the planets and their distances from the sun. Except for the mass of the moon* he approximates to the results now known with astonishing closeness. He then considers the five chief irregularities in the orbit of the moon. He further shews how the elements of a comet can be determined by three observations, and applies his results to several comets; before this time it had been believed that comets had nothing to do with the solar system.

Lastly the *Principia* is concluded by a general scholium containing reflections on the constitution of the universe, and on "the eternal, the infinite, and perfect Being" by whom it is governed.

The chief alterations in the second edition published in 1713 were the substitution of simpler proofs for some of the propositions in the second section of the first book; a more full and accurate investigation (founded on some fresh experiments made by Newton about the year 1690) of the resistance of fluids in the seventh section of the second book; and the addition of a detailed examination of the causes of the precession of the equinoxes and the theory of comets in the third book.

The chief alterations in the third edition published in 1726

* In the first edition he estimated (page 87) that the ratio of the mass of the moon to that of the earth was approximately that of 1 : 26. In the second and third editions this was altered to a ratio which is nearly that of 1 : 40.

were in the scholium on fluxions; and the addition of a new scholium on the motion of the moon's nodes (book III. prop. 53). A list of these is given in Appendix, No. xxx. to the second volume of Brewster's life of Newton.

The *Optics* was published in 1704, and contains a statement of Newton's theory of physical optics. It is divided into three books. The first two were written in 1675 and 1677, and the chief results in them (except the last proposition) are taken from the papers published in the *Philosophical Transactions* for 1675. The third book and the last proposition of the second book are founded on the paper in the *Philosophical Transactions* for 1687 (see p. 291), and are mainly devoted to a detailed consideration of diffraction. The subject of mathematical physics is outside the scope of this work and I have nothing to add to what I have stated above as to the general idea of the hypothesis advocated by Newton.

To this book were appended two minor works, which have no special connection with optics; one on curves of the third degree, and the other on the quadrature of curves. Both of these were old manuscripts which had long been familiar to his friends and pupils, but they were here published *urbi et orbi* for the first time. I will take them in the above order.

There is nothing to indicate exactly at what time Newton wrote the essay entitled *On curves of the third degree*, but some of it was probably composed before 1676 as he alludes to cubic curves in his letter to Leibnitz which is dated Oct. 24 of that year. The object of the paper seems to be to illustrate the use of analytical geometry, and as the application to conics was well known Newton selected the theory of cubics.

He begins with some general theorems, and classifies curves according as to whether their equations are algebraical or transcendental: the former being cut by a straight line in a number of points (real or imaginary) equal to the

degree of the curve: the latter being cut by a straight line in an infinite number of points. Newton then shews that many of the most important properties of conics have their analogues in the theory of cubics; of this he gives numerous illustrations. He next proceeds to discuss the theory of asymptotes and curvilinear diameters to curves of any degree.

After these general theorems he commences his detailed examination of cubics by pointing out that a cubic must have at least one real asymptotic direction. If the asymptote corresponding to this direction is at a finite distance it may be taken for the axis of y. This asymptote will cut the curve in three points altogether, of which at least two are at infinity. If the third point is at a finite distance then (by one of his general theorems on asymptotes) the equation can be written in the form

$$xy^2 + hy = ax^3 + bx^2 + cx + d$$

where the axes of x and y are the asymptotes of the hyperbola which is the locus of the middle points of all chords drawn parallel to the axis of y. While if the third point in which this asymptote cuts the curve is also at infinity the equation can be written in the form

$$xy = ax^3 + bx^2 + cx + d.$$

Next he takes the case where the asymptote corresponding to the real asymptotic direction is not at a finite distance. A line parallel to it may be taken as the axis of y. Any such line will cut the curve in three points altogether, of which one is by hypothesis at infinity, and one is necessarily at a finite distance. He then shews that if the remaining point in which this line cuts the curve is at a finite distance the equation can be written in the form

$$y^2 = ax^3 + bx^2 + cx + d.$$

While if it is at an infinite distance the equation can be written in the form

$$y = ax^3 + bx^2 + cx + d.$$

Any cubic is therefore reducible to one of four charac-

teristic forms. Each of these forms is then discussed in detail, and the possibility of the existence of double points, isolated ovals, &c. is thoroughly worked out. The final result is that there are in all seventy-two possible forms which a cubic may take. To these Stirling* in his *Lineæ tertii ordinis Newtonianae* published in 1717 added four; and Cramer† and Murdoch‡ in the *Genesis curvarum per umbras* published in 1746 each added one; thus making in all seventy-eight species.

In the course of the analysis Newton states the remarkable theorem that in the same way as the conics may be considered as the shadows of a circle (i.e. plane sections of a cone on a circular base) so all cubics may be considered as the shadows of the curves represented by the equation $y^2 = ax^3 + bx^2 + cx + d$. It was thirty years before any mathematician succeeded in proving this. It was first effected by Clairaut in 1731, but the best proof is that due to Murdoch whose work is mentioned in the last paragraph. His analysis depends on the classification of these curves into five species according as to whether the points of intersection with the axis of x are real and unequal, real and two of them equal (two cases), real and all equal, or finally two imaginary and one real.

The second appendix to the *Optics* was entitled *On the quadrature of curves.* Most of it had been communicated to

* *James Stirling* was born in 1696 and died in 1770. Besides his commentary on Newton's cubic curves which was published in 1717, he wrote the *Methodus differentialis* ......, Rome, 1730. The latter work is divided into two parts; one on the summation of series of certain forms, and the other on methods of interpolation.

† *Gabriel Cramer*, born at Geneva in 1704 and died at Bagnols in 1752, was professor at Geneva. He edited the works of John Bernoulli: he also wrote on algebraic curves, on elementary determinants (1750), and on the physical cause of the spheroidal shape of the planets and the motion of their apses (1730).

‡ *Patrick Murdoch*, born in London about 1715 and died there in 1774, wrote several memoirs (most of which were published in the *Phil. Trans.*) on points in astronomy, optics, and trigonometry.

Barrow in 1666, and was probably familiar to Newton's pupils and friends from about 1667 onwards. It consists of two parts.

The bulk of the first part had been included in the letter to Leibnitz of Oct. 24, 1676, and in a statement of Newton's method of effecting the quadrature and rectification of curves by means of infinite series as described above. This part contains the earliest use of literal indices, and the first printed statement of the binomial theorem: these are however introduced incidentally. The main object of this part is to give rules for developing a function of x in a series in ascending powers of x; so as to enable mathematicians to effect the quadrature of any curve in which the ordinate y can be expressed as an explicit function of the abscissa x. Wallis had shown how this quadrature could be found when y was given as a sum of a number of powers of x (see p. 297), and Newton here extends this by showing how any function can be expressed as an infinite series in that way. I should add that Newton is generally careful to state whether the series are convergent. In this way he effects the quadrature of the curves

$$y = \frac{a^2}{b+x}, \quad y = (a^2 \pm x^2)^{\frac{1}{2}}, \quad y = (x \pm x^2)^{\frac{1}{2}}, \quad y = \left(\frac{1+ax^2}{1-bx^2}\right)^{\frac{1}{2}},$$

but the results are of course expressed as infinite series.

He then proceeds to curves whose ordinate is given as an implicit function of the abscissa; and he gives a method by which y can be expressed as an infinite series in ascending powers of x, but the application of the rule to any curve demands in general such complicated numerical calculations as to render it of little value.

He concludes this part by showing that the rectification of a curve can be effected in a somewhat similar way. His process is equivalent to finding the integral with regard to x of $(1+\dot{y}^2)^{\frac{1}{2}}$.

The second part of this work is a statement of Newton's *theory of fluxions and fluents* with numerous examples.

Without following Newton's order very closely the following is the substance of his exposition.

The idea of a fluxion, as its name indicates, is derived from that of motion. Newton states that all geometrical magnitudes may be conceived as generated by continuous motion: thus a line may be considered as generated by the motion of a point, a surface by that of a line, a solid by that of a surface, &c.; and the velocity of the moving magnitude is defined as the fluxion of the magnitude generated.

Further, if we conceive a point as moving along a curve which is referred to coordinate axes then the velocity of the moving point can be resolved into two velocities, one parallel to the axis of x, the other to that of y; these velocities are called the fluxions of x and y respectively, just as the velocity of the point is called the fluxion of the arc. Reversing the process the arc is called the fluent of the velocity with which it is described, the abscissa is the fluent of the component velocity parallel to Ox, &c.

Newton next remarks that if the velocity of a point describing a curve be regarded as constant, then the ratio of the fluxions of the abscissa and ordinate of any point on it will depend only on the nature of the curve. Conversely the equation of the curve can be determined from the relation which exists at each instant between the fluxions of the coordinates of a point describing it. The rules to solve the first part of the problem form the "method of fluxions"; and those to solve the second and converse part form the "inverse method of fluxions."

He further observes that not only do the coordinates of a point on a curve change, but also the subtangent, normal, radius of curvature, &c.; all these quantities accordingly have fluxions, whose ratios are determined by the motion of the point, and conversely these quantities may themselves be regarded as fluents. Similar remarks apply to areas and surfaces.

Newton however points out that though he has defined these magnitudes as if they were functions of the time he

does not consider the time as necessarily entering into his problems, as it is sufficient to suppose that one of the proposed quantities to which the others are referred increases equably. This quantity or fluent may be chosen at pleasure, and is what we now are accustomed to call the independent variable.

Returning to the second or inverse problem of fluxions Newton goes on to say that it involves three cases. *First* when the given equation or condition contains the fluxions of two quantities and only one of their fluents. The simplest and most common case of this class is what is now known as integration and arises when the fluxion (whose fluent is required) is directly given. In Newton's time this was usually termed the method of quadratures, for it is the same as the problem of finding the area of a curve (since the fluxion of an area when the abscissa is taken as the principal fluent is the ordinate). The *second* class is when the given equation involves both the fluents and the fluxions. This problem is therefore the same as the solution of a differential equation; this was what Newton called the inverse method of tangents. The *third* class is when the given equation involves the fluents and the fluxions of three or more quantities. This problem is therefore the same as that of the solution of a partial differential equation. Newton solves no problems of this class and I presume they were beyond his powers of analysis.

These are the fundamental principles of his method. It now remains to describe the notation he introduced. If any quantities (regarded as fluents) be represented by letters, such as x, y, z, &c. the corresponding fluxions are represented by $\dot{x}$, $\dot{y}$, $\dot{z}$, &c. Again, if $\dot{x}$, $\dot{y}$, $\dot{z}$, &c. be regarded as variable or fluent quantities, their fluxions are represented by $\ddot{x}$, $\ddot{y}$, $\ddot{z}$, &c.; and are the fluxions of the fluxions of x, y, z, &c., that is they are the second fluxions of x, y, z, &c. Similarly we may have fluxions of the third or higher orders. If one of the quantities, x for instance, be taken as the "principal fluxion" i.e. so as to vary directly as the time then $\dot{x}$ is a constant, and consequently $\ddot{x} = 0$.

Next, $\dot{x}$, $\dot{y}$, $\dot{z}$, &c. may be regarded as themselves the fluxions of other quantities called their fluents. These fluents are in some places represented by Newton by x', y', z', &c., in other places by $[x]$, $[y]$, $[z]$, &c.; from them again we may get their fluents, i.e. the second fluents of the original quantities; and so on.

The infinitely small parts by which the variables increase in an indefinitely small time are called by Newton the "moments" of the fluents. If the time be the independent variable so that an infinitely small portion of time or moment is denoted by o, then the moments or infinitely small increments of x, y, &c., are represented by $\dot{x}o$, $\dot{y}o$, &c.; thus if x, y, &c., denote the value of the fluents at any instant, their values at the end of an indefinitely small interval of time are represented by $x+\dot{x}o$, $y+\dot{y}o$, &c.

This, as Newton observes, furnishes a ready method of drawing the tangent at any point on a curve, and it is in fact equivalent to Barrow's method already considered. He adds the important remark that thus we may in any problem neglect the terms multiplied by the second and higher powers of o, and we can always find an equation between the coordinates x, y of a point on a curve and their fluxions $\dot{x}$, $\dot{y}$. It is an application of this principle which constitutes one of the chief values of the calculus. For if we desire to find the effect produced by several causes on a system, then if we can find the effect produced by each cause when acting alone in a very small time, the total effect produced in that time will be equal to the sum of the separate effects: but I do not think that Newton realized this fact.

Such was the method of fluxions and fluents as devised by Newton in his earliest papers. It is interesting as being the form that the infinitesimal calculus first took, and Newton's treatment of it is very similar to that which is now usual. A great deal of confusion has been caused by the English writers in the eighteenth century who tried to alter the nomenclature, calling the infinitesimal increment a fluxion and denoting it by $\dot{x}$.

There is no occasion for me to fill my pages with any account of Newton's application of fluxions to various problems, chiefly on geometry, which are contained in this work. Most of them do not differ in principle from the examples which are to be found in any modern text book.

The notation of the fluxional calculus is for most purposes less convenient than that of the differential calculus. The latter was invented by Leibnitz. It was used by him in his note-books as early as 1675, and occurs in his letter to Newton in 1677. It was published in 1684. But the question whether the general idea of the calculus expressed in that notation was obtained by Leibnitz from Newton or whether it was invented independently gave rise to a long and bitter controversy.

There is no question that Newton used the method of fluxions as early as 1666, and that an account of it was communicated in manuscript to friends and pupils from and after 1669, but no description of it (other than what might be gathered from the *Principia*) was printed till 1693 some nine years after Leibnitz's account of his differential calculus had been published. Unless therefore a charge of bad faith can be established against Leibnitz he is certainly entitled to the credit of having independently invented it, and in such a matter the presumption must be in favour of his good faith. Unfortunately Leibnitz's good faith in the matter is open to question.

The facts are very briefly these. In 1704 Leibnitz wrote an anonymous review of Newton's tract on quadrature in which he made some remarks on Newton's method for which it is admitted there was no authority or justification; and amongst other statements implied that Newton had borrowed the idea of fluxional calculus from him. This review, which was correctly attributed to Leibnitz, excited considerable indignation, and led to an examination of the whole question. Till this time the statement of Leibnitz that he had discovered the calculus later than Newton but independently had been generally accepted without examination. On now looking into the matter more closely this was doubted, and in 1708 John

Keill, the Savilian professor at Oxford (born at Edinburgh in 1671 and died at Oxford in 1721), publicly accused Leibnitz of having derived the fundamental ideas of his calculus from papers by Newton which had been communicated to him through Collins and Oldenburg, and having only changed the notation and the name*. After an acrimonious controversy Leibnitz appealed to the Royal Society to compel Keill to withdraw the accusation. Newton now investigated the matter himself. There is no doubt that he was convinced that the charge was true; and on April 5, 1711, he made a speech to the society giving a complete history of the affair. A letter from Keill dated May 24, written to Leibnitz by order of the society, is an abstract of it. Leibnitz in his reply on Dec. 29, 1711, asked the society to adjudicate the matter; and a committee was accordingly appointed to go into it. They reported on April 24, 1712, and decided that Keill's charge was substantiated. This report is known as the *Commercium Epistolicum*: an analysis of it drawn up by Newton was published in the transactions of the society in 1715. Leibnitz was not represented before the committee, and they had no opportunity of hearing any explanation he could have offered: it will therefore be safer to put their decision entirely on one side, and treat it as an *ex parte* statement of Newton's case. The *Commercium Epistolicum* has been critically examined by de Morgan in the *Companion to the Almanack* and the *Philosophical Magazine* for 1852, and by Biot and Lefort in an edition of it which was published in Paris in 1856. These writers agree in saying that it shows a marked bias in favour

* As a matter of fact a similar charge against Leibnitz had been made a few years earlier, in 1699, by Duillier (1664—1753). Leibnitz at once denied the truth of it in his journal, the *Acta eruditorum*, for May, 1700; and he cited Newton's remarks in the *Principia* as practically admitting that they had discovered the calculus independently, though the scholium in question hardly seems to bear this interpretation. The editors refused to publish Duillier's reply in which he tried to substantiate his case; but Duillier was a person of little importance, and his statements did not excite much attention at the time they were made.

The death of Leibnitz in 1716 only put a temporary stop to the controversy which was bitterly debated for many years later. The question depends on circumstantial evidence of what took place more than two hundred years ago, and it is now hardly feasible to demonstrate the truth or falsity of the charge. It is however impossible to defend Leibnitz's conduct in the controversy: his duplicity, his alteration of two of Bernoulli's letters and interpolations in them, his anonymous and unjust attacks on his opponents, his reckless charges of bad faith against Newton which he did not attempt to substantiate, and his constant efforts to import other matters into the controversy, do not affect the primary question at issue, but they do seriously weaken his case which is almost entirely based on the presumption of his honour in the matter.

If we turn to the evidence itself it will be noticed that the case against Leibnitz rests chiefly on the fact that when he came to London in 1673 and 1676 he had only recently begun to turn his attention to research in mathematics. He admitted in a private letter to Conti written shortly before his death that Collins had in 1676 shown him some of the Newton correspondence, but implied that it was of little or no value. Now seeing that he discussed the question of analysis by infinite series with Collins and Oldenburg both in 1673 and in 1676, it is *à priori* probable that they would have shown him the manuscript of Newton on that subject (which forms the tract on quadrature ultimately published in 1704), a copy of which was possessed by one or both of them and of which the results were well known in London at the time. Again Leibnitz received Newton's letter of 1676, and it would be strange if he had not availed himself of the source of information there disclosed (if only to see whether the method was different from his own) unless he was already aware of the results. Lastly the letter to Newton in 1677 shows that Leibnitz was then in possession of the differential method, and apparently in as complete a form as that in which he

published it in 1684: now Collins* died in 1683 and Oldenburg in 1678, so that the publication by Leibnitz of his discoveries was immediately after the death of the only two witnesses who knew the whole truth. Such is the case against Leibnitz. The case for him rests entirely on the presumption of his honour, and is admirably stated by Biot and Lefort.

For myself I think that in this correspondence from 1700 to 1716 Leibnitz did honestly believe he had invented the subject independently of Newton; but he was writing from memory of what had happened nearly forty years before, and I suspect that the manuscripts of Newton which he privately admitted that he saw in 1676 may have been seen in 1673, and were far more important than he recollected later: indeed to a man of his ability a few hints would have given him the clue how to approach the problems he was then attacking. After all the question is one of evidence, and every one can form for themselves the opinion which on the whole seems to be most probable. The matter occupies a place in the history of mathematics which is quite disproportionate to its true importance.

If we must confine ourselves to one system of notation then there can be no doubt that that which was invented by Leibnitz is better fitted for most of the purposes to which the infinitesimal calculus is applied than that of fluxions, and for some (such as the calculus of variations) it is indeed almost essential. It should however be remembered that at the beginning of the eighteenth century the methods of the infinitesimal calculus had not been systematized, and either

* *John Collins*, whose name so frequently occurs in this controversy, was born near Oxford on March 5, 1625 and died in London on Nov. 10, 1683. He was a man of great natural ability but of slight education; being devoted to mathematics he spent all his spare time in correspondence with the leading mathematicians of the time, for whom he was always ready to do anything in his power. To him we are indebted for much information on the details of the discoveries of the period. See Rigaud's *Correspondence of scientific men of the seventeenth century*, 2nd edition with additions by de Morgan, Oxford, 1862.

notation was equally good. The development of that calculus was the main work of the mathematicians of the first half of the eighteenth century. The application of it by Euler, Lagrange, and Laplace to the principles of mechanics laid down in the *Principia* was the great achievement of the last half of that century, and finally demonstrated the superiority of the differential to the fluxional calculus. The translation of the *Principia* into the language of modern analysis and the filling in of those details of the Newtonian theory by the aid of that analysis was effected by Laplace.

The controversy with Leibnitz was regarded in England as an attempt by foreigners to defraud Newton of the credit of his invention, and the question was complicated on both sides by national jealousies. It was therefore natural though it was unfortunate that the geometrical and fluxional methods as used by Newton were alone studied and employed at Cambridge. The consequence was that in spite of the brilliant band of scholars formed by Newton the improvement of the method of analysis was almost wholly effected on the continent; and it was not until about 1820 that under the influence of Babbage, Peacock, and Herschel (see p. 408) the value of the differential calculus was recognized at Cambridge, and that Newton's countrymen again took any large share in the development of physical astronomy.

The remaining mathematical works of Newton are the *Universal Arithmetic*, the *Lectiones opticae*, the *Methodus differentialis*, and the *Analytical geometry*.

The *Universal Arithmetic* is a copy of a manuscript which had been written about 1669 or 1670, and had continued to circulate in the university in much the same way as the numerous mathematical manuscripts containing matter which has not yet got incorporated into text-books do at the present time. Whiston* who succeeded Newton in the Lucasian chair

* *William Whiston*, born in Leicestershire on Dec. 9, 1667, educated at Clare College, Cambridge, of which society he was a fellow, and died

extracted a somewhat reluctant permission from Newton to print it, and it was published in 1707.

The work contains a large number of algebraical and geometrical problems. The problems are interesting, and the solutions needless to say extremely elegant. Amongst several new theorems on various points in algebra and the theory of equations the following important results were here first enunciated. Newton explained that the equation whose roots are the solution of a given problem will have as many roots as there are different possible cases, and he also considered how it happened that the equation to which a problem led might contain roots which did not satisfy the original question. He used the principle of continuity to explain how two real and unequal roots might become imaginary in passing through equality, and illustrated this by geometrical considerations; thence he showed that imaginary roots must occur in pairs. Newton also here gave rules to find a superior limit to the positive roots of a numerical equation, and to determine the approximate values of the numerical roots. He further enunciated the theorem known by his name for finding the sum of the nth powers of the roots of an equation, and laid the foundation of the theory of symmetrical functions of the roots of an equation.

Perhaps the most interesting theorem contained in the work is his attempt to find a rule (analogous to that of Descartes for real roots) by which the number of imaginary roots of an equation can be determined. He knew that the result which he obtained was not universally true, but he gave no proof and did not explain what were the exceptions to the rule. His theorem is as follows. Suppose the equation to

in London on Aug. 22, 1752, wrote several works on astronomy. He acted as Newton's deputy in the Lucasian chair from 1699, and in 1703 succeeded him as professor, but he was expelled in 1711 because he had asserted that he could not understand the mystery of the Trinity. He was succeeded by Nicholas Saunderson, the blind mathematician, who was born in Yorkshire in 1682 and died at Christ's College, Cambridge, on April 19, 1739.

be of the nth degree arranged in descending powers of x (the coefficient of x^n being positive), and suppose the $n+1$ fractions

$$1,\quad \frac{n}{n-1}\,\frac{2}{1},\quad \frac{n-1}{n-2}\,\frac{3}{2},\ \ldots\ \frac{n-p+1}{n-p}\,\frac{p+1}{p}\ \ldots\ \frac{2}{1}\,\frac{n}{n-1},\quad 1$$

to be formed and written below the corresponding terms of the equation, then if the square of any term when multiplied by the corresponding fraction is greater than the product of the terms on each side of it put a plus sign above it: otherwise put a minus sign above it, and put a plus sign above the first and last terms. Now consider any two consecutive terms in the original equation, and the two symbols written above them. Then we may have any one of the four following cases: (α) the terms of the same sign and the symbols of the same sign; (β) the terms of the same sign and the symbols of opposite signs; (γ) the terms of opposite signs and the symbols of the same sign; (δ) the terms of opposite signs and the symbols of opposite signs. Then it has been shown that the number of negative roots will not exceed the number of cases (α), and the number of positive roots will not exceed the number of cases (γ); and therefore the number of imaginary roots is not less than the number of cases (β) and (δ). In other words the number of changes of signs in the row of symbols written above the equation is an inferior limit to the number of imaginary roots. Newton however asserted that "You may almost know how many roots are impossible" by counting the changes of sign in the series of symbols formed as above. That is to say he thought that in general the actual number of positive, negative and imaginary roots could be got by the rule and not merely superior or inferior limits to these numbers. But though he knew that the rule was not universal he could not find what were the exceptions to it: this theorem was subsequently discussed by Campbell, Maclaurin, Euler, and other writers; at last in 1864 Prof. Sylvester succeeded in proving the general result (see the *Phil. Trans.* for April, 1864, and the *Phil. Mag.* for March, 1866).

The work entitled *On analysis by infinite series* was published in 1711 and is practically the same as the first part of the tract on quadrature described above on p. 324. It is said that this was originally intended to be an appendix to Kinckhuysen's algebra (see p. 288).

The *lectiones opticae* published in 1729 have been already mentioned on p. 289 and consist of the lectures on geometrical optics delivered at Cambridge in the years from 1669 to 1671. The chief results are contained in the papers published in the *Philosophical Transactions* from 1671 to 1676. The work is divided into two books, the first of which contains four sections and the second five. The first section of the first book deals with the decomposition of solar light by a prism in consequence of the unequal refrangibility of the rays that compose it, and gives a full account of his experiments. The second section contains an account of the method which Newton invented for determining the coefficients of refraction of different bodies. This is done by making a ray pass through a prism of the material so that the angle of incidence is equal to the angle of emergence; he shews that if the angle of the prism is i and the total deviation of the ray is δ the refractive index is

$$\sin \tfrac{1}{2}(i+\delta) \operatorname{cosec} \tfrac{1}{2} i.$$

The third section is on refractions at plane surfaces. Most of this section is devoted to geometrical solutions of different problems, many of which are very difficult. He here finds the condition that a ray may pass through a prism with minimum deviation. The fourth section treats of refractions at curved surfaces. The second book treats of his theory of colours and of the rainbow.

The tract entitled *Methodus differentialis* was published in 1736 and contains an account of Newton's method of interpolation. The principle is this. If $y = \phi(x)$ is a function of x and if when x is successively put equal to $a_1, a_2, \ldots$ the values of y are known and are $b_1, b_2, \ldots$ then a parabola whose equation is

$$y = p + qx + rx^2 + \ldots$$

can be drawn through the points (a_1, b_1), (a_2, b_2), ... and the ordinate of this parabola may be taken as an approximation to the ordinate of the curve. The degree of the parabola will of course be one less than the number of given points. Newton points out that in this way the area of any curve can be approximately determined.

The *Analytical geometry* was probably written as a manuscript for his pupils between 1670 and 1680; but it too was not published till 1736, some years after Newton's death. Besides a statement of his methods of quadrature and rectification and of finding fluxions and fluents, which are similar to those published in the *Optics* in 1704, he treats of fluxional (or differential) equations and he considers the application of the fluxional calculus to geometry. The latter part contains rules for determining the equation of a tangent and the radius of curvature at any point of a curve, the points of inflexion on a curve, and other similar problems. He also investigates the rule for finding the maximum and minimum values of functions of one variable and obtains the same result as that now in use. We regard the change of sign of the difference between two consecutive values of the function as the true criterion: but his argument is that when a quantity increasing has attained its maximum it can have no further increment; or when decreasing it has attained its minimum it can have no further decrement; consequently the fluxion must be equal to nothing.

Besides these works, extracts from his books and summaries of them were published; but with these Newton himself seems to have had nothing to do.

CHAPTER XVII.

LEIBNITZ AND THE MATHEMATICIANS OF THE FIRST HALF OF THE EIGHTEENTH CENTURY.

SECTION 1. *Leibnitz and the Bernoullis.*
SECTION 2. *The development of analysis on the continent.*
SECTION 3. *The English mathematicians of the eighteenth century.*

I HAVE briefly traced in the last chapter the nature and extent of Newton's contributions to science. Modern analysis is however derived directly from the works of Leibnitz and the elder Bernoullis; and it is immaterial to us whether the fundamental ideas of it were obtained by them from Newton, or discovered independently. The English mathematicians of the years considered in this chapter continued to use the language and notation of Newton; they are thus somewhat distinct from their continental contemporaries, and I have therefore grouped them together in a section by themselves.

Leibnitz and the Bernoullis.

Gottfried Wilhelm **Leibnitz*** was born at Leipzig on June 21 (O. S.), 1646 and died at Hanover on Nov. 14, 1716. His father died before he was six, and the teaching at the school to which he was then sent was inefficient, but his industry triumphed over all difficulties; by the time he was twelve he

* See the life of Leibnitz by G. E. Guhrauer, 2 vols. and a supplement, Breslau, 1842 and 1846; a more concise account is given in the memoir of Leibnitz by F. Kirchner, Heidelberg, 1876.

had taught himself to read Latin easily, and had begun Greek; and before he was twenty he had mastered all the ordinary text-books on mathematics, philosophy, theology, and law. Refused the degree of doctor of laws at Leipzig by those who were jealous of his youth and learning he moved to Nuremberg. An essay which he there wrote on the study of law was dedicated to the elector of Mainz, and led to his appointment by the elector on a commission for the revision of some statutes, from which he was subsequently promoted to the diplomatic service. In the latter capacity he supported (unsuccessfully) the claims of the German candidate for the crown of Poland. The violent seizure of various small places in Alsace in 1670 excited universal alarm in Germany as to the designs of Louis XIV.; and Leibnitz drew up a scheme by which it was proposed to offer German co-operation if France liked to take Egypt, and use the possession of that country as a basis for attack against Holland in Asia, while Germany itself was to be left undisturbed by France. This bears a curious resemblance to the similar plan by which Napoleon I. proposed to attack England. In 1672 Leibnitz went to Paris on the invitation of the French government to explain the details of the scheme, but nothing came of it.

At Paris he met Huygens who was then residing there, and their meetings led him to study geometry, which he described as opening a new world to him, though he had as a matter of fact previously written some tracts on various minor points in mathematics; the most important of them being a paper on combinations written in 1668, and a description of a new calculating machine. In January, 1673, he was sent on a political mission to London, where he stopped some months and made the acquaintance of Oldenburg, Collins, and others; it was at this time that he communicated the memoir to the Royal Society in which he was found to have been forestalled by Mouton (see p. 292).

In 1673 the elector of Mainz died, and in the following year Leibnitz entered the service of the Brunswick family; in 1676

he again visited London, and then moved to Hanover, where till his death he had charge of the ducal library, and received a handsome stipend. His pen was thenceforth employed in all the political matters which affected the Hanoverian family, and his services were recognized by honours and distinctions of various kinds: his memoranda on the various political, historical, and theological questions which concerned the dynasty during the forty years from 1673 to 1713 form a valuable contribution to the history of that time. His appointment in the Hanoverian service gave him increased leisure for his favorite pursuits. Leibnitz used to assert that as the first-fruit of his increased leisure he invented the differential and integral calculus in 1674, but the earliest traces of the use of it in his extant note-books do not occur till 1675, and it was not till 1677 that we find it developed into a consistent system. (It was not published till 1684.) Nearly all his mathematical papers were produced within the ten years from 1682 to 1692, and most of them in a journal called the *Acta eruditorum* which he had founded in 1678, and which had a very wide circulation on the continent. All those hereafter alluded to were published in this journal.

In 1700 the Academy of Berlin was created on his advice, and he drew up the first body of statutes for it. On the accession in 1714 of his master George I. to the throne of England Leibnitz was practically thrown aside as a useless tool; he was forbidden to come to England; and the last two years of his life were spent in neglect and dishonour. He died at Hanover in 1716. He was over fond both of money and personal distinctions; but he possessed singularly attractive manners, and all who once came under the charm of his personal presence remained sincerely attached to him.

Leibnitz occupies at least as large a place in the history of philosophy as he does in the history of mathematics. Most of his philosophical writings were composed in the last twenty or twenty-five years of his life; and curiously enough the question as to whether his views were original or whether they were

appropriated from Spinoza, whom he visited in 1676, is still in question among philosophers; though the evidence seems to me to point to the originality of Leibnitz. Some fresh correspondence between them is said to have been discovered in the summer of 1888, and is now being edited by Dr Stein of Zurich. As to his system of philosophy it will be enough to say that he regarded the ultimate elements of the universe as individual percipient beings whom he called monads. According to him the monads are centres of force, and substance is force, while space, matter, and motion are merely phenomenal: finally the existence of God is inferred from the existing harmony among the monads. His services to literature were almost as considerable as those to philosophy, but it is his mathematical work alone that concerns me here.

All his mathematical papers have been collected and edited by C. J. Gerhardt in 6 vols., Berlin 1849—1860. The chief subjects discussed in them are the infinitesimal calculus and some mechanical problems.

The only papers of first-rate importance which he produced are those on the differential calculus. The earliest of these was one published in the *Acta eruditorum* for October 1684 in which he enunciated a general method for finding maxima and minima, and for drawing tangents to curves. One inverse problem, namely to find the curve whose subtangent is constant, was also discussed. The notation is the same as that with which we are familiar, and the differential coefficients of x^n and of products and quotients are determined. In 1686 he wrote a paper on the principles of the new calculus. In both of these papers the principle of continuity is explicitly assumed, while his treatment of the subject is based on the use of infinitesimals and not on that of the limiting value of ratios. In answer to some objections which were raised in 1694 by Bernard Nieuwentyt (born at Westgraafdyke in 1654 and died at Purmerende in 1718) who asserted that $\frac{dy}{dx}$ stood for an unmeaning quantity like $\frac{0}{0}$,

Leibnitz explained that the value of $\frac{dy}{dx}$ in geometry could be expressed as the ratio of two finite quantities, in the same way as Barrow had previously done. I do not think that Leibnitz's statement of the objects and methods of the infinitesimal calculus as contained in these papers, which are the three most important memoirs on it that he produced, are as able as those given by Newton and quoted above, and his attempt to place the subject on a metaphysical basis did not tend to clearness; but the notation he introduced is superior to that of Newton, and the fact that all the results of modern mathematics are expressed in the language invented by Leibnitz has proved the best monument to his work.

In 1686 and 1692 he wrote papers on osculating curves. These however contain some bad blunders; as for example, the assertion that an osculating circle will necessarily cut a curve in four consecutive points: this error was pointed out by John Bernouilli, but in his article of 1692 Leibnitz defended his original assertion, and insisted that a circle could never cross a curve where it touched it.

In 1692 Leibnitz wrote a memoir in which he laid the foundation of the theory of envelopes. This was further developed in another paper in 1694, in which he introduced for the first time the terms coordinates and axes of coordinates.

Leibnitz also published a good many papers on mechanical subjects; but some of them contain mistakes which show that he did not understand the principles of the subject. Thus in 1685 he wrote a memoir to find the pressure exerted by a sphere of weight W placed between two inclined planes of complementary inclinations (which he supposes placed so that the lines of greatest slope are perpendicular to the line of the intersection of the planes). He asserts that the pressure on each plane must consist of two components "unum quo decliviter descendere tendit, alterum quo planum declive premit." He further says that "for metaphysical reasons" the

sum of the two pressures must be equal to W. Hence, if R and R' be the required pressures, and a and $\frac{1}{2}\pi - a$ the inclinations of the planes, he finds that

$$R = \tfrac{1}{2} W (1 - \sin a + \cos a) \text{ and } R' = \tfrac{1}{2} W (1 - \cos a + \sin a).$$

The true values are $R = W \cos a$ and $R' = W \sin a$.

Nevertheless some of his papers on mechanics are valuable. Of these the most important were two in 1689 and 1694, in which he solved the problem of finding an isochronous curve; one in 1697, on the curve of quickest descent (this was the problem sent as a challenge to Newton); and two in 1691 and 1692, in which he stated the intrinsic equation of the curve taken by a flexible rope suspended from two points, i.e. the catenary, but gave no proof. This last problem had been originally proposed by Galileo.

In 1689, that is two years after the *Principia* had been published, he wrote on the movements of the planets which he stated were produced by a motion of the ether. Not only were the equations of motion which he obtained wrong, but his deductions from them were not even in accordance with his own axioms. In another memoir in 1706, that is nearly twenty years after the *Principia* had been written, he admitted that he had made some mistakes in his former paper but adhered to his previous conclusions, and sums the matter up by saying "it is certain that gravitation generates a new force at each instant to the centre, but the centrifugal force also generates another away from the centre....The centrifugal force may be considered in two aspects according as the movement is considered as along the tangent to the curve or along the arc of the circle itself." It seems clear from this paper that he did not really understand the manner in which Newton had reduced dynamics to an exact science. It is hardly necessary to consider his work on dynamics in further detail. Much of it is vitiated by a constant confusion between momentum and kinetic energy. He sometimes uses the first which he calls the *vis mortua*, and sometimes the latter the

double of which he calls the *vis viva*, as the measure of a force; according as the force is "passive" or "active."

The series quoted by Leibnitz comprise those for e^x, $\log(1+x)$, $\sin x$, $\operatorname{vers} x$, and $\tan^{-1}x$. All of these had been previously published, and he rarely if ever added any demonstrations. In 1693 he explained the method of expansion by indeterminate coefficients, though his applications were not free from error.

To sum the matter up briefly, it seems to me that Leibnitz's work exhibits great skill in analysis; but wherever he leaves his symbols and attempts to interpret his results or deal with concrete cases he commits blunders; and on the whole I think his mathematical work is overrated. No doubt the demands of politics and philosophy on his time may have prevented him from elaborating any subject completely or writing any systematic exposition of his views; but they are no excuse for the mistakes of principle which occur so frequently in his papers. Some of his memoirs contain suggestions of methods which have now become valuable means of analysis, such as the use of determinants and indeterminate coefficients. But when a writer of manifold interests like Leibnitz throws out innumerable suggestions, some of them are likely to turn out valuable; and to enumerate these (which he never worked out) without reckoning the others which are wrong seems to me to give a wholly false impression of the value of his work.

Leibnitz was only one amongst several continental writers whose papers in the *Acta eruditorum* familiarized mathematicians with the use of the differential calculus. The most important of these were Jacob and John Bernoulli, both of whom were warm friends and admirers of Leibnitz, and to whose unselfish devotion his reputation is largely due. Not only did they take a prominent part in nearly every mathematical question then discussed, but nearly all the leading mathematicians on the continent for the first half of the eighteenth century came directly or indirectly under the influence of the teaching of one or both of them.

The Bernouillis (or as they are sometimes called the Bernoullis) were a family of Dutch origin, who were driven from Holland by the Spanish persecutions and finally settled at Bâle in Switzerland. The first member of the family who attained any marked distinction in mathematics was Jacob. *Jacob* Bernouilli, usually called James Bernouilli by English writers, was born at Bâle on Dec. 27, 1654 and died there on Aug. 16, 1705. He was one of the earliest to realize how powerful as an instrument of analysis was the differential calculus, and he applied it to several problems, but he did not himself invent any new processes. His most important discoveries were his solution of the problem to find an isochronous curve; his proof that the construction for the catenary which had been given by Leibnitz was correct, and his extension of this to strings of variable density and under a central force; his determination of the form taken by an elastic rod fixed at one end and acted on by a given force at the other, the *elastica*; also of a flexible rectangular sheet with two sides fixed horizontally and filled with a heavy liquid, the *lintearia*; and lastly of a sail filled with wind, the *volaria*. In 1696 he offered a reward for the general solution of isoperimetrical figures, i.e. the determination of a figure of a given species which should include a maximum area, its perimeter being given: his own solution published in 1701 is substantially correct. In 1698 he published an essay on the differential calculus which contains numerous applications to geometry. He here investigated the chief properties of the equiangular spiral and especially noticed the manner in which various curves deduced from it reproduced the original curve: struck by this fact he begged that in imitation of Archimedes (see p. 60) an equiangular spiral should be engraved on his tombstone with the inscription *eadem mutata resurgo*. He also brought out an edition of Descartes' *Geometry*; and in his *Ars conjectandi* published in 1713 he established the fundamental principles of the calculus of probabilities. His works were collected and published in three volumes; one issued at Bâle in 1713, and two at Geneva in 1744.

Johann Bernouilli, the brother of the preceding, was born at Bâle on Aug. 7, 1667, and died there on Jan. 1, 1748. He filled the chairs of mathematics successively at Groningen and Bâle. To all who did not acknowledge his merits in a manner commensurate with his own view of their importance he behaved most unjustly: as an illustration of his character it may be mentioned that he attempted to substitute for an incorrect solution of his own on isoperimetrical curves another stolen from his brother Jacob, while he expelled his son Daniel from his house for obtaining a prize from the French Academy which he had expected to receive himself. He was however the most successful teacher of his age, and had the faculty of inspiring his pupils with almost as passionate a zeal for mathematics as he felt himself. His great influence was uniformly and successfully exerted in favour of the use of the differential calculus, and his lessons on it, which were written in 1691 and are published in vol. III. of his works, shew how completely he had even then grasped the principles of the new analysis. These lectures, which contain the earliest use of the term integral, were the first attempt to construct an integral calculus; for Newton and Leibnitz had treated each problem by itself, and neither of them had laid down any general rules on the subject. Leaving out of account his innumerable controversies, the chief discoveries of John Bernouilli were the exponential calculus, the treatment of trigonometry as a branch of analysis, the conditions for a geodesic, the determination of orthogonal trajectories, the solution of the brachistochrone, the statement that a ray of light traversed such a path that $\Sigma\mu ds$ was a minimum, and the enunciation of the principle of virtual work. I believe that he was the first to denote the accelerating effect of gravity by an algebraical sign g, and he thus arrived at the formula $v^2 = 2gh$: the same result would have been previously expressed by the proportion $v_1^2 : v_2^2 = h_1 : h_2$. His works were published by Cramer at Geneva in four volumes in 1745.

Several members of the same family but of a younger generation enriched mathematics by their teaching and writings.

The most important of these were the three sons of John; namely, Nicholas, Daniel, and John the younger; and the two sons of John the younger, who also bore the names of John and Jacob. To make the account complete I add here their respective dates. *Nicholas* Bernouilli, the eldest of the three sons of John, was born in 1695 and was drowned at St Petersburg where he was professor in 1726. *Daniel* Bernouilli, the second son of John, was born on Feb. 9, 1700 and died in 1782. He was professor first at St Petersburg and afterwards at Bâle, and shares with Euler the unique distinction of having gained the prize proposed annually by the French Academy no less than ten times. His earliest mathematical essay was a solution given in 1724 of the differential equation proposed by Riccati. His chief work was on hydrodynamics and was published in 1738. The solutions of the problem of vibrating cords which had been given by Taylor (see p. 357) and d'Alembert were discussed by him and Euler. *Johann* Bernouilli, the younger, a brother of Nicholas and Daniel, was born on May 18, 1710 and died in 1790. He also was a professor at Bâle. He left two sons John and Jacob: of these, *Johann*, who was born on Dec. 4, 1744 and died on July 10, 1807, was astronomer royal and director of mathematical studies at Berlin: and his brother *Jacob*, who was born on Oct. 17, 1759 and died in July, 1789, was successively professor at Bâle, Verona, and St Petersburg.

The development of analysis on the continent.

Leaving for a moment the English mathematicians of this half of the eighteenth century we come next to a number of continental writers who barely escape mediocrity and to whom it will only be necessary to devote a few words. Their writings mark the steps by which analytical geometry, and the differential and integral calculus, were perfected and made familiar to mathematicians. Nearly all of them were pupils of one or other of the two elder Bernouillis, and they were so nearly contemporaries that it is difficult to arrange them chronologically. The most eminent of them are *de Gua*: *de Mont-*

mort: *Fagnano*: *l'Hospital*: *Nicole*: *Parent*: *Riccati*: *Saurin*; and *Varignon*. I will take them as far as possible in their order of time.

Guillaume François Antoine l'Hospital, Marquis de St-Mesme, born at Paris in 1661 and died there on Feb. 2, 1704, was among the earliest pupils of John Bernoulli. He took part in most of the challenges issued by Leibnitz, the Bernoullis, and other continental mathematicians of the time; in particular he gave a solution of the brachistochrone, and investigated the form of the solid of least resistance of which Newton in the *Principia* had stated the result. He wrote in 1696 a treatise on the differential calculus which did a great deal to make its advantages widely known in France; the only new work in this is his investigation of the limiting value of the ratio of functions which for a certain value of the variable take the indeterminate form 0 : 0. A supplement to this, containing a similar treatment of the integral calculus, together with the additions to the differential calculus which had been made in the following half century, was published at Paris, 1754—6, by L. A. de Bougainville. The marquis de l'Hospital also wrote a treatise on analytical conics which was published in 1707.

Antoine Parent, born at Paris on Sept. 16, 1666 and died there on Sept. 26, 1716, was the first to refer a surface to three coordinate planes and thus determine its form by means of an equation: this was in 1700. His works were collected and published in 3 vols., Paris, 1713.

Pierre Varignon, born at Caen in 1654 and died in Paris on Dec. 22, 1722, was an intimate friend of Leibnitz and the Bernoullis, and the earliest and most powerful advocate in France of the differential calculus. He simplified the proofs of many of the leading propositions in mechanics, and recast the treatment of the subject. His works were published at Paris in 1725.

Joseph Saurin, born at Courtaison in 1659 and died at Paris on Dec. 29, 1737, was the first to show how the tangents at the multiple points of curves could be always determined.

François Nicole, who was born at Paris on Dec. 23, 1683 and died there on Jan. 18, 1758, was the first to publish a systematic treatise on finite differences. Taylor had regarded the differential coefficient, i. e. the ratio of two infinitesimal differences, as the limiting value of the ratio of two finite differences, a method which is still used by many English writers, though it has been generally abandoned on the continent, and had thus been led to give a sketch of the subject in his *Methodus* published in 1715 (see p. 357). Nicole's *Traité du calcul des différences finies* was published in 1717. It is a well-arranged and able book, and contains rules both for forming differences and effecting the summation of given series. Besides this in 1706 he wrote a work on roulettes, especially spherical epicycloids: and in 1731 he published a memoir on Newton's essay on curves of the third degree.

Pierre Raymond de Montmort, born at Paris on Oct. 27, 1678 and died there on Oct. 7, 1719, was also interested in the subject of finite differences. He determined in 1718 the sum of p terms of a finite series of the form

$$na + \frac{n(n-1)}{1 \cdot 2} \Delta a + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \Delta^2 a + \dots .$$

Jean Paul de Gua was born at Carcassonne in 1713 and died at Paris on June 2, 1785. He published in 1740 a work on analytical geometry in which he applied it, without the aid of the differential calculus, to find the tangents, asymptotes, and various singular points of an algebraical curve; and he further showed how singular points and isolated loops were affected by conical projection. He gave the proof of Descartes' rule of signs which is to be found in all modern works: it is not clear whether Descartes ever proved it strictly, and Newton seems to have regarded it as obvious.

Jacopo Francesco Count Riccati, born at Venice on May 28, 1676 and died at Trèves on April 15, 1754, did a great deal to disseminate a knowledge of the Newtonian philosophy in Italy. Besides the equation known by his name, certain cases

of which he succeeded in integrating, he discussed the question of the possibility of lowering the order of a given differential equation. His works were published at Trèves in 4 vols. in 1758. He had two sons who wrote on several minor points connected with the integral calculus and differential equations, and applied the calculus to several mechanical questions: these were *Vincenzo*, who was born in 1707 and died in 1775, and *Giordano*, who was born in 1709 and died in 1790.

Giulio Carlo Count de Fagnano, born at Sinigaglia on Dec. 6, 1682 and died on Sept. 26, 1766, may be said to have been the first writer who directed attention to the theory of elliptic functions. Failing to rectify the ellipse or hyperbola Fagnano attempted to determine arcs whose difference should be rectifiable. He also pointed out the remarkable analogy existing between the integrals which represent the arc of a circle and the arc of a lemniscate. Finally he proved the formula

$$\pi = 8i \log \frac{1-i}{1+i},$$

where i stands for $\sqrt{-1}$. His works were collected and published in 2 vols. at Pesaro in 1750.

It was inevitable that some mathematicians should object to the methods of analysis by means of the infinitesimal calculus. The most prominent of these were *Viviani*, *de Lahire*, and *Rolle*. Chronologically they come here but they flourished half a century after the date to which their writings properly belong.

Vincenzo Viviani, a pupil of Galileo and Torricelli, born at Florence on April 5, 1622 and died there on Sept. 22, 1703, brought out a restoration of the lost book of Apollonius on conic sections in 1659; and a restoration of the work of Aristæus in 1701. He explained in 1677 how an angle could be trisected by the aid of the equilateral hyperbola or the conchoid. In 1692 he proposed the problem to construct four windows in a hemispherical vault so that the remainder of

the surface can be accurately determined, a problem which attracted the attention of Wallis, Leibnitz, David Gregory, and Jacob Bernouilli.

Philippe **de Lahire**, born in Paris on March 18, 1640 and died there on April 21, 1718, wrote on graphical methods, 1673; on the conic sections, 1685; a treatise on epicycloids, 1694; one on roulettes, 1702; and lastly another on conchoids, 1708. His works on conic sections and epicycloids were founded on the teaching of Desargues, whose favorite pupil he was. He also translated the essay of Moschopulus on magic squares, and collected together all the theorems on them which were previously known: this was published in 1705.

Michel **Rolle**, born at Ambert on April 21, 1652 and died in Paris on Nov. 8, 1719, wrote an algebra in 1689 which contains the theorem on the position of the roots of an equation which is known by his name. He published in 1696 a treatise on the solution of equations whether determinate or indeterminate, and he produced several other minor works. He taught that the differential calculus was nothing but a collection of ingenious fallacies.

So far no one of the school of Leibnitz and the two Bernouillis had shown any exceptional ability, but by the action of a number of second-rate writers the methods and language of analytical geometry and the differential calculus were everywhere well known by about 1740. The close of this school is marked by the appearance of *Clairaut* and *d'Alembert*. Their lives overlap the period considered in the next chapter, but though it is difficult to draw a sharp dividing line which shall separate by a definite date the mathematicians there considered from those that form the subject of this chapter I think that on the whole their works are best treated here.

Alexis Claude **Clairaut** was born at Paris on May 13, 1713 and died there on May 17, 1765. He belongs to the small group of children who though of exceptional precocity survive

and maintain their powers when grown up. As early as the age of twelve he wrote a memoir on four geometrical curves, but his first important work was his treatise on tortuous curves published when he was eighteen—a work which procured for him immediate admission to the French Academy. In 1731 he gave a demonstration of the fact noted by Newton that all curves of the third order were projections of one of five parabolas. In 1741 he was sent to measure the length of a meridian degree on the earth's surface, and on his return in 1743 he published his *Théorie de la figure de la Terre*. This is founded on a paper by Maclaurin, where it had been shewn that a mass of homogeneous fluid set in rotation about a line through its centre of mass would, under the mutual attraction of its particles, take the form of a spheroid. This work of Clairaut treated of heterogeneous spheroids and contains the proof of his formula for the accelerating effect of gravity in a place of latitude l namely

$$g = G\left\{1 - \left(\tfrac{5}{2}m - \epsilon\right)\left(\tfrac{1}{3} - \cos^2 l\right)\right\},$$

where G is the value of equatorial gravity, m the ratio of the centrifugal force to gravity at the equator, and ϵ the ellipticity of a meridian section of the earth. In 1849 Prof. Stokes showed in the *Camb. Phil. Trans.* vol. VIII. that the same result was true whatever was the density of the earth provided the surface was a level one.

Impressed by the power of geometry as shown in the writings of Newton and Maclaurin, Clairaut abandoned analysis and his next work, the *Théorie de la Lune*, published in 1752, is strictly Newtonian in character. This contains the explanation of the motion of the apse which had previously puzzled astronomers (see p. 315), and which Clairaut had at first deemed so inexplicable that he was on the point of publishing a new hypothesis as to the law of attraction when it occurred to him to carry the approximation to the third order, and he thereupon found that the result was in accordance with the observations. This was followed in 1754 by some lunar tables;

and Clairaut subsequently wrote several papers on the orbit of the moon, and the motion of comets, particularly on the path of Halley's comet.

His growing popularity in society hindered his scientific work: "engagé," says Bossut, "à des soupers, à des veilles, entraîné par un goût vif pour les femmes, voulant allier le plaisir à ses travaux ordinaires, il perdit le repos, la santé, enfin la vie à l'âge de cinquante-deux ans."

Jean-le-Rond D'Alembert was born at Paris on Nov. 16, 1717 and died there on Oct. 29, 1783. He was the illegitimate child of the chevalier Destouches. Being abandoned by his mother on the steps of the little church of St Jean-le-Rond which then nestled under the great porch of Nôtre Dame, he was taken to the parish commissary, who following the usual practice in such cases gave him the christian name of Jean-le-Rond: I do not know by what title he subsequently assumed the right to prefix *de* to his name. He was boarded out by the parish with the wife of a glazier in a small way of business who lived near the cathedral, and here he seems to have found a real home though a very humble one. His father appears to have looked after him, and paid for his going to a school where he obtained a fair mathematical education. An essay on the integral calculus written in 1738, and another on "ducks and drakes" or ricochets in 1740 attracted considerable attention, and in the same year he was elected a member of the French Academy; this was probably partly due to the influence of his father, but it is to his credit that he absolutely refused to leave his adopted parents.

Nearly all his mathematical works were produced within the next ten years. The first of these was his *Traité de dynamique*, published in 1743, in which he enunciates the principle known by his name (see p. 313): the application of this principle enables us to write the differential equations of motion of any rigid system. In 1744 he published his *Traité de l'équilibre et du mouvement des fluides*, in which he applies his equations to fluids: this led to partial differential equations

which he was then unable to solve. In 1745 he developed that part of the subject which dealt with the motion of air in his *Théorie générale des vents*: and this again led him to partial differential equations. A second edition of this in 1746 was dedicated to Frederick the Great of Prussia and procured an invitation to Berlin and the offer of a pension: he declined the former, but after some pressing pocketed his pride and the latter. In 1747 he applied the differential calculus to the problem of a vibrating string, and again arrived at a partial differential equation of the form

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}.$$

His analysis had thus three times brought him to the same equation; and he at last succeeded in showing that it was satisfied by

$$u = \phi(x+t) + \psi(x-t),$$

where ϕ and ψ are arbitrary functions.

This is the most brilliant piece of analysis he produced, and it may be interesting to give his solution which was published in the transactions of the Berlin Academy for 1747. He begins by saying that if $\dfrac{\partial u}{\partial x}$ be denoted by p and $\dfrac{\partial u}{\partial t}$ by q, then

$$du = pdx + qdt.$$

But by the given equation $\dfrac{\partial p}{\partial x} = \dfrac{\partial q}{\partial t}$, and therefore $pdt + qdx$ is also an exact differential: denote it by dv.

Therefore $$dv = pdt + qdx.$$

Hence

$$du + dv = (pdx + qdt) + (pdt + qdx) = (p+q)(dx+dt),$$

and $$du - dv = (pdx + qdt) - (pdt + qdx) = (p-q)(dx-dt).$$

Thus $u+v$ must be a function of $x+t$, and $u-v$ must be a function of $x-t$. We may therefore put

$$u + v = 2\phi(x+t),$$

and $$u - v = 2\psi(x-t).$$

Hence $$u = \phi(x+t) + \psi(x-t).$$

D'Alembert added that the conditions of the physical problem of a vibrating string demanded that when $x=0$ u should vanish for all values of t. Hence identically

$$\phi(t)+\psi(-t)=0.$$

Assuming that both functions could be expanded in integral powers of t, this required that they should only contain odd powers. Hence

$$\psi(-t)=-\phi(t)=\phi(-t).$$

Therefore $$u=\phi(x+t)+\phi(x-t).$$

Euler now took the matter up and shewed that the equation of the form of the string was $\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$, and that the general integral was $u=\phi(x-at)+\psi(x+at)$, where ϕ and ψ are arbitrary functions. Most of Euler's results on the theory of partial differential equations are collected in his *Institutiones calculi integralis.*

The chief remaining contributions of d'Alembert to mathematics were on physical astronomy, especially on the precession of the equinoxes and on variations in the obliquity of the ecliptic. These were collected in his *Système du monde* published in 3 vols. in 1754. An edition of all his works was published at Paris in 8 vols., 1761—1780.

During the latter part of his life he was mainly occupied with the great French encyclopædia. For this he wrote the introduction, and numerous philosophical and mathematical articles: the best are those on geometry and on probabilities.

The English mathematicians of the eighteenth century.

I have reserved a notice of the English mathematicians who succeeded Newton in order that the members of the English school may be all treated together. It was almost a matter of course that the English should at first have adopted the notation of Newton in the infinitesimal calculus in pre-

ference to that of Leibnitz, and the English school would consequently in any case have developed on somewhat different lines to that on the continent, where a knowledge of the infinitesimal calculus was derived solely from Leibnitz and the Bernoullis. But this separation into two distinct schools became very marked owing to the action of John Bernoulli, who regarded the controversy on the origin of the infinitesimal calculus as a convenient opportunity to vent his dislike of Newton and Newton's countrymen. It was only natural though it was unfortunate that the English should have resented this by declining to see any merit in the works of Leibnitz and John Bernoulli: and so for forty or fifty years to the mutual disadvantage of both sides the quarrel raged. The leading members of the English school were Cotes, Taylor, de Moivre and Maclaurin. The following is an alphabetical list of those here considered: *Cotes, de Moivre, David Gregory, Landen, Maclaurin, Robert Smith, Stewart,* and *Taylor.*

David **Gregory**, the nephew of the James Gregory mentioned on p. 278, born at Aberdeen on June 24, 1661 and died at Maidenhead on Oct. 10, 1708, was appointed professor at Edinburgh in 1684, and in 1691 was on Newton's recommendation elected Savilian professor at Oxford. His chief works are a Geometry issued in 1684; an Optics in 1695; and a treatise on the Newtonian geometry, physics, and astronomy in 1702. It was to him that Newton sent the manuscript of the optical lectures given at Cambridge in the years from 1669 to 1671, and from which the edition of 1729 was printed.

Brook **Taylor**, born at Edmonton on Aug. 18, 1685 and died in London on Dec. 29, 1731, was educated at St John's College, Cambridge, and was among the most enthusiastic of Newton's admirers. From the year 1708 onwards he wrote numerous papers in the *Philosophical Transactions* in which among other things he discussed the motion of projectiles, the centre of oscillation, and the forms of liquids raised by capillarity*. His earliest work was one on perspective published

* This was first considered by *Humphry Ditton*, who was born at

in 1715; but that by which he is generally known is his *Methodus incrementorum directa et inversa* published in London, 1715—1717. This contains the proof of the well-known theorem

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{\underline{|2}} f''(x) + \dots,$$

by which any function of a single variable can be expanded in powers of it. He does not consider the convergency of the series, and the proof which contains numerous assumptions is not worth reproducing. Taylor was the earliest writer to deal with theorems on the change of the independent variable: he was also the first to realize the possibility of a calculus of operations; and just as he denotes the nth differential coefficient of y by y_n, so he uses y_{-1} to represent the integral of y. The applications of the calculus to various questions which Taylor gave in the *Methodus* have hardly received that attention they deserve. The most important of them is his theory of the transverse vibrations of strings, a problem which had baffled all previous investigators. In this he finally determines that the number of half-vibrations executed in a second is

$$\pi \sqrt{\frac{DP}{LN}},$$

where L is the length of the string, N its weight, P the weight which stretches it, and D the length of a seconds pendulum. This is correct, but in arriving at it he assumed that every point of the string would pass through its position of equilibrium at the same instant, a restriction which d'Alembert subsequently showed to be unnecessary. Taylor also found the form which the string assumes at any instant. This work also contained the earliest determination of the differential equation of the path of a ray of light when traversing a heterogeneous medium; and assuming that the density of the air depended only on its distance from the earth's surface

Salisbury in 1675, died in 1715. Ditton was the author of numerous mathematical works.

Taylor obtained by means of quadratures the approximate form of the curve.

Roger Cotes was born near Leicester on July 10, 1682 and died at Cambridge on June 5, 1716. He was educated at Trinity College, Cambridge, of which society he was a fellow, and in 1706 was elected to the newly created Plumian chair of astronomy in the university of Cambridge. From 1709 to 1713 his time was almost wholly occupied in editing the second edition of the *Principia*. The remark of Newton that if only Cotes had lived "we should have learnt something" indicates the opinion of his abilities held by most of his contemporaries. Cotes' writings were collected and published in 1722 under the title *Harmonia mensurarum*. A large part of this work is given up to the decomposition and integration of rational algebraical expressions: that part which deals with the theory of partial fractions was left unfinished, but was completed by de Moivre. His theorem in trigonometry which depends on forming the quadratic factors of $x^n - 1$ is well known. The proposition that "if from a fixed point O a line be drawn cutting a curve in $Q_1, Q_2, \ldots Q_n$, and a point P be taken on it so that the reciprocal of OP is the arithmetic mean of the reciprocals of $OQ_1, OQ_2, \ldots OQ_n$, then the locus of P will be a straight line" is also due to Cotes. The title of the book was derived from the latter theorem.

Robert Smith, born in 1689 and died at Cambridge on Feb. 2, 1768, was also educated at Trinity College, Cambridge, of which society he was a fellow, and subsequently master. He was a cousin of Cotes, whose works he edited and whom he succeeded as Plumian professor. His *Opticks* published in 1728 is one of the best text-books on the subject that has yet appeared, and with a few additions might be usefully reprinted now. He also published in 1760 a work on sound entitled *Harmonics*.

Abraham de Moivre was born at Vitry on May 26, 1667 and died in London on Nov. 27, 1754. His parents came to England when he was a boy; and his education and friends were alike English. He taught mathematics in London, and

was intimately connected with Newton, Halley, and other mathematicians of the English school. The manner of his death has a curious interest for psychologists. Shortly before it, he declared that it was necessary for him to sleep some ten minutes or a quarter of an hour longer each day than the preceding one: the day after he had thus reached a total of something over twenty-three hours he slept up to the limit of twenty-four hours, and then died in his sleep.

He is most celebrated for having, together with Lambert, created that part of trigonometry which deals with imaginary quantities. Two theorems on this part of the subject are still connected with his name: namely that which asserts that $\sin nx + i \cos nx$ is one of the values of $(\sin x + i \cos x)^n$, and that which gives the various quadratic factors of $x^{2n} - 2px^n + 1$. His chief works, other than numerous papers in the *Philosophical Transactions*, were *The doctrine of chances* published in 1716, and the *Miscellanea analytica* published in 1730. In the former the theory of recurring series was first given, and the theory of partial fractions which Cotes' premature death had left unfinished was completed, while the rule for finding the probability of a compound event was enunciated. The latter contains some theorems in astronomy besides the trigonometrical propositions mentioned above.

Colin **Maclaurin**, who was born at Kilmodan in Argyllshire in February 1698 and died at York on June 14, 1746, was educated at the university of Glasgow, 1709—1714; in 1717, he was appointed at the early age of nineteen professor of mathematics at Aberdeen; and in 1725, he succeeded James Gregory in his chair at Edinburgh. Maclaurin took an active part in opposing the advance of the Young Pretender in 1745: on the approach of the Highlanders he fled to York, but the exposure in the trenches at Edinburgh and the privations he endured in his escape proved fatal to him. He competed twice, in 1717 and 1724, for the annual prize offered by the French Academy, and in each case obtained it.

His chief works are his *Geometrica organica*, London, 1719;

his *De linearum geometricarum*, London, 1720; his *Treatise on fluxions*, Edinburgh, 1742; his *Account of Newton's discoveries*, London, 1748; and his *Algebra*, London, 1748.

The *Geometrica organica* is on the extension of a theorem given by Newton. Newton had shewn that if two angles bounded by straight lines turn round their respective summits so that the point of intersection of two of these lines moves along a straight line the other point of intersection will describe a conic; and if the first moves along a conic the second will describe a quartic. Maclaurin gave an analytical discussion of the general theorem, and showed how by this method various curves could be practically traced. This work contains an elaborate discussion on curves and their pedals, a branch of geometry which he had created in two papers in the *Phil. Trans.* for 1718 and 1719.

In the following year, 1720, Maclaurin issued a supplement which is practically the same as his *De linearum geometricarum*. It is divided into three sections and an appendix. The first section contains a proof of Cotes' theorem above alluded to; and also the analogous theorem (discovered by himself) that "if a straight line OP_1P_2 ... drawn through a fixed point O cut a curve of the nth degree in n points P_1, P_2, ... and if the tangents at P_1, P_2, ... cut a fixed line Ox in points A_1, A_2, ... then the sum of the reciprocals of the distances OA_1, OA_2, ... will be constant for all positions of the line OP_1P_2 ...." These two theorems are generalizations of those given by Newton on diameters and asymptotes. Either is deducible from the other. In the second section these theorems are applied to conics; most of the harmonic properties connected with an inscribed quadrilateral are determined; and in particular the theorem on an inscribed hexagon which is known by the name of Pascal is deduced. Pascal's essay was not published till 1779 and the earliest printed enunciation of his theorem was that given by Maclaurin. In the third section these theorems are applied to cubic curves. Amongst other propositions he here shews that if a quadrilateral be inscribed in a cubic, and

if the points of intersection of the opposite sides also lie on the curve, then the tangents to the cubic at any two opposite angles of the quadrilateral will meet on the curve. The appendix contains some general theorems. One of these (which includes Pascal's as a particular case) is that "if a polygon be deformed so that while each of its sides passes through a fixed point, its angles (save one) describe respectively curves of the mth, nth, pth,... degrees, then shall the remaining angle describe a curve of the degree $2mnp$...; but if the given points are collinear, the resulting curve will only be of the degree mnp ... ." This essay was reprinted with additions in the *Phil. Trans.* for 1735.

The *Treatise of fluxions* published in 1742 was the first systematic exposition of this method. To the pure calculus he added the theorem that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\underline{|2}} f''(0) + \dots .$$

This was obtained in the manner given in most modern text-books by assuming that $f(x)$ can be expanded in a form like

$$f(x) = A_0 + A_1 x + A_2 x^2 + \dots ,$$

then on differentiating and putting $x = 0$ in the successive results, the values of A_0, A_1, ... are obtained: but he did not investigate the convergency of the series. Maclaurin also gave the correct theory of maxima and minima, and rules for finding and discriminating multiple points.

This treatise is however especially valuable for the solutions it contains of numerous problems in geometry, statics, the theory of attractions, and astronomy. To solve these he reverted to classical methods, and so powerful did these processes seem when used by him that Clairaut after reading the work abandoned analysis, and attacked the problem of the figure of the earth again by pure geometry. At a later time this part of the book was described by Lagrange as the "chef-d'œuvre de géométrie qu'on peut comparer à tout ce qu'Archi-

mède nous a laissé de plus beau et de plus ingénieux" (*Mém. de l'Acad. de Berlin*, 1773). Maclaurin also determined the attraction of a homogeneous ellipsoid at an internal point, and gave some theorems on its attraction at an external point; in effecting this he introduced the conception of level surfaces, i.e. surfaces at every point of which the resultant attraction is perpendicular to the surface. No further advance in the theory of attractions was made until Legendre took up the subject in 1783 (see p. 392). Maclaurin also shewed that a spheroid was a possible form of equilibrium of a mass of homogeneous liquid rotating about an axis passing through its centre of mass. Finally he discussed the tides; this part had been previously published (in 1740) and had received the prize from the French Academy.

Among Maclaurin's minor works is his *Algebra* published in 1748, and founded on Newton's *Universal Arithmetic*. It contains the results of some early papers of Maclaurin; notably of two written in 1726 and 1729 on the number of imaginary roots of an equation, suggested by Newton's theorem (see p. 334); and of one written in 1729 containing the well-known rule for finding equal roots by means of the derived equation. To this a treatise entitled *De linearum geometricarum proprietatibus generalibus* was added as an appendix. It is the same as the paper of 1720 above alluded to, but contains some additional theorems of great elegance. Maclaurin also produced in 1728 an exposition of the Newtonian philosophy, but this was not printed till 1748. Almost the last paper he wrote was one printed in the *Phil. Trans.* for 1743 in which he discussed the form of a bee's cell from a mathematical point of view.

Maclaurin was one of the most able mathematicians of the eighteenth century, but his influence on the progress of British mathematics was on the whole unfortunate. By himself abandoning the use both of analysis and of the infinitesimal calculus he induced Newton's countrymen to confine themselves to Newton's methods, and as I remarked before it was not until 1817 when the differential calculus was introduced

into the Cambridge curriculum that English mathematicians made any general use of the more powerful methods of modern analysis.

Almost the only other British writer of any marked eminence in pure mathematics during the eighteenth century was *Matthew* Stewart who succeeded Maclaurin in his chair at Edinburgh. Stewart was born at Rothsay in 1717 and died at Edinburgh on Jan. 23, 1785. He studied under Simson in Glasgow (see foot-note p. 49) and subsequently under Maclaurin. Stewart's chief works were *General theorems...*, Edinburgh, 1746; *Tracts physical and mathematical*, London, 1761; *Propositiones geometricae more veterum demonstratae*, Edinburgh, 1763; and *A solution of Kepler's problem*, Edinb. Phil. Soc. 1771. These prove him to have been a mathematician of great natural power, but unfortunately he followed the fashion set by Newton and Maclaurin, and confined himself to geometrical methods. The *General theorems* contain many of the results of modern geometry as applied to the circle and straight line, and most of the elementary properties of transversals and involution. His theorems on the problem of three bodies and other questions of physical astronomy are singularly ingenious. Matthew Stewart was in his turn succeeded in 1775 as professor of mathematics by his son Dugald Stewart, the celebrated philosopher.

Another English mathematician of the same date, whose writings served as starting points for the researches of others, was *John* Landen who was born near Peterborough on Jan. 23, 1719 and died at Milton on Jan. 15, 1790. Euler and Lagrange commenced their discussion of elliptic integrals by considering his theorem published in 1755 connecting the arcs of a hyperbola and an ellipse; while Lagrange's *Calcul des fonctions* is based on the ideas contained in his *Residual analysis* published in 1764. His writings on pure mathematics are suggestive rather than powerful; those on applied mathematics are worthless.

CHAPTER XVIII.

LAGRANGE, LAPLACE, AND THEIR CONTEMPORARIES, CIRC. 1740—1830.

SECTION 1. *The development of analysis and mechanics.*
SECTION 2. *The creation of modern geometry.*
SECTION 3. *The development of mathematical physics.*
SECTION 4. *The introduction of analysis into England.*

I HAVE indicated in chapter XVI. the nature of the revolution in mathematics effected by Newton's work between the years 1666 and 1686. We may say that it was not till about the year 1740 that his discoveries were thoroughly understood and assimilated by mathematicians, and a considerable part of the last chapter deals with those who helped to make them appreciated. We come now to mathematicians who began to build on the foundation he had laid.

Pre-eminent among the subjects considered by Newton were the infinitesimal calculus, mechanics, universal gravitation, and optics. The first of these had been extended by the labours of Taylor and Maclaurin in England, but the fluxional notation which they used was inconvenient for many purposes. On the continent under the influence of John Bernoulli the calculus had become an instrument of great analytical power expressed in an admirable notation—and for practical applications it is impossible to over-estimate the value of a good notation—but the continental school had confined themselves almost entirely to obtaining a thorough knowledge of the differential and

integral calculus without considering the uses to which it could be put. The subject of mechanics remained very much in the condition in which Newton had left it, until d'Alembert in putting Newton's results into the language of the differential calculus did something to extend it. Universal gravitation as enunciated in the *Principia* was accepted as an established fact, but the geometrical methods adopted in proving it were difficult to follow, or to use in analogous problems; Maclaurin and Clairaut may be regarded as the last mathematicians of any distinction who employed them. Lastly the Newtonian theory of light was generally received as correct.

The leading mathematicians of the era on which we are now entering are Euler, Lagrange, Legendre, and Laplace. Briefly we may say that Euler extended, summed up, and completed the work of his predecessors; while Lagrange with almost unrivalled skill developed the infinitesimal calculus and theoretical mechanics into the form in which we now know them. At the same time Laplace made some additions to the infinitesimal calculus, and applied that calculus to the theory of universal gravitation; he also created a calculus of probabilities. Legendre invented spherical harmonic analysis, and elliptic integrals; and added to the theory of numbers. The works of these writers are still standard authorities and are hardly yet the subject-matter of history. I shall therefore content myself with a mere list of their chief discoveries, referring any one who wishes to know more to the works themselves. Lagrange, Laplace, and Legendre created a French school of mathematics of which the younger members are divided into two groups; one of which (including Poisson and Fourier) began to apply mathematical analysis to physics, and the other (including Monge, Carnot, and Poncelet) created modern geometry. Strictly speaking some of the great mathematicians of recent times, such as Gauss and Abel, were contemporaries of the above; but their analysis is of a different character, and except for this remark I defer any consideration of them to the next chapter.

The development of analysis and mechanics.

Leonhard **Euler** was born at Bâle on April 15, 1707 and died at St Petersburg on Sept. 7, 1783. He was the son of a lutheran minister who had settled at Bâle, and was educated in his native town under the direction of John Bernouilli, with whose sons Daniel and Nicholas he formed a life-long friendship. When the younger Bernouillis went in 1725 to Russia, on the invitation of the empress, they procured a place there for Euler, which in 1733 he exchanged for the chair of mathematics then vacated by Daniel Bernouilli. The severity of the climate affected his eyesight and in 1735 he lost the use of one eye completely. In 1741 he moved to Berlin at the request or rather command of Frederick the Great: here he stayed till 1766, when he returned to Russia, and was succeeded at Berlin by Lagrange. Within two or three years of his going back to St Petersburg he became blind; but in spite of this, and although his house together with many of his papers were burnt in 1777, he recast and improved most of his earlier works. He died of apoplexy in 1783. He was married twice.

I think we may sum up Euler's work by saying that he revised the analysis of all the parts of pure mathematics which were then known, filled up the details, added proofs, and arranged the whole in a consistent form. Such work is very important and it is fortunate for science when it falls into hands as competent as those of Euler.

Euler wrote an immense number of memoirs on all kinds of mathematical subjects, the mere enumeration of which in Fuss's* éloge occupies 51 pages. His chief works are as follows. In the first place he wrote in 1748 his *Introductio in analysin infinitorum*, which was intended to serve as an introduction to pure analytical mathematics. This is divided into

* *Nicholas Fuss* born at Bâle in 1755 and died at St Petersburg in 1826, was a pupil of Daniel Bernouilli, and subsequently was appointed assistant to Euler. He wrote on spherical conics and on lines of curvature.

two parts. The first contains the bulk of the matter which is to be found in modern text-books on algebra, theory of equations, and trigonometry. In this he paid particular attention to the expansion of various functions in series, and to the summation of given series; and for the first time we find the rule laid down that an infinite series cannot safely be employed unless it is convergent. In his trigonometry, much of which is founded on C. Mayer's *Arithmetic of sines* published in 1727, he developed the idea of John Bernoulli that the subject was a branch of analysis and not a mere appendage of astronomy or geometry: he also introduced the current abbreviations* for the trigonometrical functions, and showed the connection between the trigonometrical and exponential functions. Here too we first meet the symbol π used to denote the incommensurable number 3·14159 .... This quantity like that denoted by the base of Napierian logarithms would enter into mathematical analysis from whatever side the subject was approached: it represents among other things the ratio of the circumference of a circle to its diameter, but it is a mere accident that that is taken for its definition. De Morgan in the *Budget of paradoxes* tells an anecdote which illustrates how misleading such a definition might be. He was explaining to an actuary what was the chance that a certain proportion of some group of people would at the end of a given time be alive; and quoted the actuarial formula involving π, which in answer to a question he explained stood for the ratio of the circumference of a circle to its diameter. His acquaintance who had so far listened to the explanation with interest interrupted him and exclaimed, "My dear friend, that must be a delusion, what can a circle have to do with the number of people alive at the end of a given time?" The second part of the *Introductio* is on analytical geometry. Euler commenced this part by dividing curves into algebraical

* These were simultaneously introduced in England by *Thomas Simpson* (born at Bosworth in 1710 and died at Woolwich, where he was a professor, in 1761) in his *Trigonometry* published in 1748.

and transcendental, and established a variety of propositions which are true for all algebraical curves. He then applied these to the general equation of the second degree in two dimensions, showed that it represents the various conic sections, and deduced most of their properties from the general equation. He next discussed the question as to what surfaces are represented by the general equation of the second degree in three dimensions, and how they may be discriminated one from the other: some of these surfaces had not been previously investigated. In the course of this analysis he laid down the rules for the transformation of coordinates in space. Here also we find the earliest attempt to bring tortuous curves and the curvature of surfaces within the domain of mathematics.

The *Introductio* was followed in 1755 by the *Institutiones calculi differentialis* to which it was intended as an introduction. This is the first text-book on the differential calculus which has any claim to be both complete and accurate; and it may be said that all modern treatises on the subject are based on it.

This series of works was completed by the publication in three volumes in 1768 to 1770 of the *Institutiones calculi integralis* in which the results of some of Euler's memoirs on the same subject are included. This like the similar treatise on the differential calculus summed up all that was then known on the subject, but many of the theorems were recast and the proofs improved. The Beta and Gamma functions were invented by Euler and are here discussed, but only as an illustration of methods of reduction and integration. The works that form this trilogy have gone through numerous subsequent editions.

The classic problems on isoperimetrical curves, the brachistochrone in a resisting medium, and the theory of geodesics (all of which had been suggested by his master John Bernoulli) had engaged Euler's attention at an early date; and in solving them he was led to the calculus of variations. The general idea of this was laid down in his *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes* published in 1744, but the complete development of the new calculus

was first effected by Lagrange in 1759. The method used by Lagrange is described in Euler's integral calculus, and is the same as that given in any modern text-book on the subject.

In 1770 Euler published the *Anleitung zur Algebra* in two volumes. The first volume treats of determinate algebra. This contains one of the earliest attempts to place the fundamental processes on a scientific basis: the same subject had attracted d'Alembert's attention. This work also includes the proof of the binomial theorem for an unrestricted index which is still known by Euler's name; it is curious that it should have been inserted here, for not only is it not correct, but the easiest elementary proof that has yet been propounded had been published (I think in 1764) by Abnit Vandermonde (1736—1793) and must have been known to Euler who had himself pointed out the necessity of considering the convergency of an infinite series. The second volume treats of indeterminate or Diophantine algebra. This contains the solutions of some of the problems proposed by Fermat, and which had hitherto remained unsolved: in particular those mentioned above on p. 261 (*a*) and p. 262 (*e*). A French translation of the algebra with numerous and valuable additions was brought out by Lagrange in 1795; and a treatise on arithmetic by Euler was appended to it.

These four works comprise most of what Euler produced in pure mathematics. He also wrote numerous memoirs on nearly all the subjects of applied mathematics and physics: the chief results in them are as follows.

In the mechanics of a rigid system he determined the general equations of motion of a body about a fixed point, which are ordinarily written in the form

$$A\frac{d\omega_1}{dt} - (B - C)\,\omega_2\omega_3 = L:$$

and he gave the general equations of motion of a free body which are usually presented in the form

$$\frac{d}{dt}(mu) - mv\theta_3 + mw\theta_2 = X, \text{ and } \frac{dh_1'}{dt} - h_2'\theta_3 + h_3'\theta_2 = L.$$

He also defended and elaborated the theory of "least action" which had been propounded by Maupertuis (born at St Malo in 1698 and died at Bâle in 1759).

In hydrodynamics he established the general equations of motion which are commonly expressed in the form

$$\frac{1}{\rho}\frac{dp}{dx}=X-\frac{du}{dt}-u\frac{du}{dx}-v\frac{du}{dy}-w\frac{du}{dz}.$$

At the time of his death he was engaged in writing a treatise on hydromechanics in which the treatment of the subject would have been completely recast.

His most important works on astronomy are his *Theoria motuum planetarum et cometarum*, published in 1744; his *Theoria motus lunæ*, published in 1753; and his *Theoria motuum lunæ*, published in 1772. In these he attacked the problem of three bodies: he supposed the body considered, e.g. the moon, to carry three rectangular axes with it in its motion, the axes moving parallel to themselves, and to these axes all the motions were referred. This method is not convenient, but it was from Euler's results that Mayer* constructed the lunar tables for which his widow in 1770 received £5000, being the prize offered by the English parliament, and in recognition of Euler's services a sum of £300 was voted as an honorarium to him.

Euler was much interested in optics. In 1746 he discussed the relative merits of the emission and undulatory theories of light; he on the whole preferred the latter. In 1771 he published his optical researches in three volumes under the title *Dioptrica*.

He also wrote an elementary work on physics and the fundamental principles of mathematical philosophy. This originated from an invitation he received when he first went to Berlin to give lessons on physics to the princess of Anhalt-Dessau. These lectures were published in 1770 in 2 vols.

* *Tobias Mayer* born in Wurtemburg in 1723 and died in 1762, was director of the English observatory at Göttingen. Most of his memoirs, other than his lunar tables, were published in 1775 under the title *Opera inedita*.

under the title *Lettres...sur quelques sujets de physique...*, and for half a century remained the best treatise on the subject.

There is no complete edition of Euler's writings.

Of course Euler's magnificent works were not the only text-books containing original matter produced at this time. Amongst numerous writers I would specially single out *Lambert*, *Bézout*, *Landen*, *Trembley*, *Arbogaste*, and *Lhulier* as having influenced the development of mathematics. Landen has been already referred to (see p. 363).

Johann Heinrich **Lambert** * was born at Mulhouse on Aug. 28, 1728 and died at Berlin on Sept. 25, 1777. He was the son of a small tailor, and had to rely on his own efforts for his education; from a clerk in some iron-works, he got a place in a newspaper office, and subsequently on the recommendation of the editor he was appointed tutor in a private family which secured him the use of a good library and sufficient leisure to use it. In 1759 he settled at Augsburg, and in 1763 removed to Berlin where he was given a small pension and finally made editor of the Prussian astronomical almanack. His most important works were one on optics issued in 1759, which suggested to Arago the lines of investigation he subsequently pursued: a treatise on perspective in 1759 (to which in 1768 an appendix giving practical applications was added): and a treatise on comets in 1761, containing the well-known expression for the area of a focal sector of a conic in terms of the chord and the bounding radii. Besides these he communicated numerous papers to the Berlin Academy of which the most important are the following: namely, his memoir on transcendental magnitudes in 1768, in which he proved that π is incommensurable (the proof is given in Legendre's *Géometrie* and is there extended to π^2): his paper on trigonometry read in 1768, in which he developed de Moivre's theorems on the trigonometry of complex variables, and introduced the hyper-

* See *Lambert nach seinem Leben und Wirken* by D. Huber, Bâle, 1829.

bolic sine and cosine denoted by the symbols *sinh x*, *cosh x*: his essay entitled analytical observations published in 1771, which is the earliest attempt to form functional equations by expressing the given properties in the language of the differential calculus, and then integrating: lastly his paper on vis viva published in 1783, in which for the first time he expressed Newton's second law of motion in the notation of the differential calculus in the manner given above on p. 311. Most of Lambert's earlier papers are collected in his *Beiträge zum Gebrauche der Mathematik* published in 4 vols. at Berlin from 1765 to 1772.

Etienne Bézout, born at Nemours on March 31, 1730 and died on Sept. 27, 1783, besides numerous minor works wrote a *Théorie générale des équations algébriques* published at Paris in 1779 which in particular contained much new and valuable matter on the theory of elimination and symmetrical functions of the roots of an equation. He used determinants in a paper in the *Hist. de l'Acad. Roy.* 1764, but did not treat of the general theory.

Jean Trembley, born at Geneva in 1740 and died on Sept. 18, 1811, contributed to the development of differential equations, finite differences, and the calculus of probabilities.

Louis Arbogaste, born at Muntzig on Oct. 4, 1759 and died at Strassburg on April 8, 1803, wrote on series and the derivatives known by his name. He was the first writer to separate the symbols of operation from those of quantity.

Simon Antoine Jean Lhulier, born at Geneva on April 24, 1750 and died on March 28, 1840, has been already mentioned (see p. 93) for his solution of Pappus' problem. He was professor at Geneva where Sturm was one of his pupils. He wrote numerous text-books.

I do not wish to crowd my pages with an account of those who have not distinctly advanced the subject, but I have mentioned the above writers because their names are still well known. We may however practically say that the discoveries of Euler and Lagrange in the subjects which they treated were so

complete and far reaching that there was but little left for their less gifted contemporaries to add.

While discussing the mathematicians of the end of the eighteenth century I ought to mention Mascheroni's curious treatise on the geometry of the compass published at Pavia in 1795. Euclid had supposed that his readers had the use of a ruler and a pair of compasses. *Lorenzo* **Mascheroni** (who was born at Castagnota on May 14, 1750 and died at Paris on July 30, 1800) set himself the task to obtain the same results when no other construction except such as could be made with a pair of compasses was allowed. Cardan and Tartaglia had amused themselves with similar problems; but Mascheroni's work is so extraordinary a *tour de force* that it is worth chronicling. He was professor first at Bergama and afterwards at Pavia, and left numerous minor works.

Joseph Louis **Lagrange**, the greatest mathematician since the time of Newton, was born at Turin on Jan. 25, 1736 and died at Paris on April 10, 1813. His father who had the charge of the Sardinian military chest was of good social position and wealthy, but before his son grew up he had lost most of his property in speculations, and young Lagrange had to rely for his position on his own abilities. He was educated at the college of Turin, but it was not until he was seventeen that he showed any taste for mathematics: his interest in the subject being first excited by a memoir by Halley (*Phil. Trans.* vol. XVIII. p. 960) across which he came by accident. Alone and unaided he threw himself into mathematical studies, and at the end of a year's incessant toil he was already an accomplished mathematician, and was made a lecturer in the artillery school. The first fruit of these labours was his letter, written when he was still only nineteen, to Euler in which he solved the isoperimetrical problem which for more than half a century had been a constant subject of discussion. To effect the solution (in which he sought to determine the form of a function so that a formula in which it entered should satisfy

a certain condition) he enunciated the principles of the calculus of variations. Euler recognized the generality of the method adopted, and its superiority to that used by himself; and with rare and graceful courtesy he withheld a paper he had previously written, which covered some of the same ground, in order that the young Italian might have time to complete his work, and claim the undisputed invention of the new calculus. The name of this branch of analysis was suggested by Euler. This memoir at once placed Lagrange in the front rank of mathematicians then living.

In 1758 Lagrange established with the aid of his pupils a society, which was subsequently incorporated as the Turin Academy, and in the five volumes of its transactions, usually known as the *Miscellanea Taurinensia* most of his early writings are to be found. Many of these are elaborate works. The first volume contains a memoir on the theory of the propagation of sound; in this he indicates the mistake made by Newton, obtains the general differential equation for the motion, and integrates it for motion in a straight line. This volume also contains the complete solution of the problem of a string vibrating transversely; in this paper he points out a lack of generality in the solutions previously given by Taylor, d'Alembert, and Euler, and arrives at the conclusion that the form of the curve at any time t is given by the equation $y = a \sin mx \sin nt$. The article concludes with a masterly discussion of echoes, beats, and compound sounds. Other articles in this volume are on recurring series, probabilities, and the calculus of variations.

The second volume contains a long paper embodying the results of several memoirs in the first volume on the theory and notation of the calculus of variations; and he illustrates its use by deducing the principle of least action, and also by solutions of several problems in dynamics.

The third volume includes the solution of several dynamical problems by means of the calculus of variations; some papers on the integral calculus: a complete solution of Fermat's problem

given above p. 262 (g), of which Wallis had previously given an empirical solution: and the general differential equations of motion for three bodies moving under their mutual attractions.

In 1761 Lagrange stood without a rival as the foremost mathematician living; but the unceasing labour of the preceding nine years had seriously affected his health, and the doctors refused to be responsible for his reason or life unless he would take rest and exercise. Although his health was temporarily restored his nervous system never quite recovered its tone, and henceforth he constantly suffered from attacks of profound melancholy.

The next work he produced was in 1764 on the libration of the moon, and an explanation as to why the same face was always turned to the earth, a problem which he treated by the aid of virtual work. This memoir was crowned by the French Academy.

He now started to go on a visit to London, but on the way fell ill at Paris. Here he was received with the most marked honour, and it was with regret he left the brilliant society of that city to return to his provincial life at Turin. His further stay in Piedmont was however short. In 1766 Euler left Berlin for Paris, and Frederick the Great immediately wrote expressing the wish of "the greatest king in Europe" to have "the greatest mathematician in Europe" resident at his court. Lagrange accepted the offer and spent the next twenty years in Prussia, where he produced not only the long series of memoirs published in the Berlin and Turin transactions but his monumental work the *Mécanique analytique*. His residence at Berlin commenced with an unfortunate mistake. Finding most of his colleagues married, and assured by their wives that it was the only way to be happy, he married. His wife soon died, but the union was not a happy one.

Lagrange was a favorite of the king, who used frequently to discourse to him on the advantages of perfect regularity of life. The lesson went home, and thenceforth Lagrange studied his mind and body as though they were machines, and found

by experiment the exact amount of work which he was able to do without breaking down. Every night he set himself a definite task for the next day, and on completing any branch of a subject he wrote a short analysis to see what points in the demonstrations or in the subject-matter were capable of improvement. He always thought out the subject of his papers before he began to compose them, and usually wrote them straight off without a single erasure or correction.

His mental activity during these twenty years was amazing. Not only did he produce his splendid *Mécanique analytique*, but he contributed between one and two hundred papers to the Academies of Berlin, Turin, and Paris. Many of these are complete treatises, and all without exception are of the highest order of excellence. Except for a short time when he was ill he produced on an average about one memoir a month. Of these I note the following as among the most important.

First, his contributions to the fourth and fifth volumes (1766—1773) of the *Miscellanea Taurinensia*; of which the most important was the one in 1771 in which he discussed how numerous astronomical observations should be combined so as to give the most probable result. And later, his contributions to the first two volumes (1784—1785) of the transactions of the Turin Academy; to the first of which he contributed a paper on the pressure exerted by fluids in motion, and to the second an article on integration by infinite series, and the kind of problems for which it is suitable.

Most of the memoirs sent to Paris were on astronomical questions, and among these I ought particularly to mention his memoir on the Jovian system in 1766, his essay on the problem of three bodies in 1772, his work on the secular equation of the moon in 1774, and his treatise on cometary perturbations in 1778. These were all written on subjects proposed by the French Academy, and in each case the prize was awarded to him.

The greater number of his papers during this time were however contributed to the Berlin Academy. Several of the

earlier of them deal with questions on *algebra*. In particular I may mention (i) his discussion of the solution of indeterminate equations in integers (1770); with special notice of indeterminate quadratics (1769). (ii) His tract on the theory of elimination (1770). (iii) His memoirs on a general process for solving an algebraical equation of any degree (1770 and 1771); this method fails for equations of an order above the fourth, because it then involves the solution of an equation of higher dimensions than the one proposed; but it gives all the solutions of his predecessors as modifications of a single principle. He found however the complete solution of a binomial equation of any degree. (iv) Lastly in 1773 he treated of determinants of the second and third order.

Several of his early papers also deal with questions connected with the neglected but singularly fascinating subject of the *theory of numbers*. Among these are (i) his proof of Méziriac's theorem that every integer which is not a square can be expressed as the sum of either two, three, or four integral squares (1770). (ii) His proof of Wilson's theorem that if n be a prime then $\underline{|n-1} + 1$ is always a multiple of n (1771). (iii) His memoirs of 1773 and 1775 which give the demonstrations of many of the results enunciated by Fermat, in particular those indicated by the letters (c), (d), (e), (f) on p. 262. (iv) His memoir of 1777 in which he showed that the equation $x^4 = y^4 + z^4$ cannot be solved in positive integers, see p. 262 (i). (v) And lastly his method for determining the factors of numbers of the form $x^2 + ay^2$. The substance of these demonstrations will be found in the *Theory of Numbers* by P. Barlow, London, 1811.

There are also numerous articles on various points of *analytical geometry*. In two of them (in 1792 and 1793) he reduced the equations of the quadrics or conicoids to their canonical forms.

During the years from 1772 to 1785 he contributed a long series of memoirs which created the science of *differential equations*, at any rate as far as partial differential equations are concerned. I do not think that any previous writer had

done anything beyond considering equations of some particular form. A large part of those results were collected in the second edition of Euler's integral calculus which was published in 1792.

His papers on *mechanics* require no special mention as the results arrived at are embodied in the *Mécanique analytique*.

Lastly there are numerous memoirs on problems in *astronomy*. Of these the most important are the following. (i) On the attraction of ellipsoids (1773): this is founded on Maclaurin's work. (ii) That in which the motion of the nodes of a planet's orbit is determined (1774). (iii) On the stability of the planetary orbits (1776). (iv) Those in which the method of determining the orbit of a comet from three observations is completely worked out (1778 and 1783): this has not indeed proved practically available, but his system of calculating the perturbations by means of mechanical quadratures has formed the basis of all subsequent researches on the subject. (v) His determination of the secular and periodic variations of the elements of the planets (1781—1784): the upper limits assigned for these agree closely with those obtained later by Leverrier, and he proceeded as far as the knowledge then possessed of the masses of the planets permitted. (vi) Those on the method of interpolation (1783, 1792, and 1793): the part of finite differences dealing therewith is now in the same condition as that in which Lagrange left it.

Over and above these various papers, he composed his great treatise, the *Mécanique analytique*. In this he lays down the law of virtual work, and from that one fundamental principle by the aid of the calculus of variations deduces the whole of mechanics, both of solids and fluids. The object of the book is to shew that the subject is implicitly included in a single principle, and to give general formulæ from which any particular result can be obtained. The method of generalized coordinates by which he obtained this result is perhaps the most brilliant result of his analysis. Instead of following the motion of each individual part of a material system, as d'Alembert and Euler had done, he shewed that if we deter-

mine its configuration by a sufficient number of variables, whose number is the same as that of the degrees of freedom possessed by the system, then the kinetic and potential energies of the system can be expressed in terms of these, and the differential equations of motion thence deduced by simple differentiation. For example, in dynamics of a rigid system he replaces the consideration of the particular problem by the general equation which is now usually written in the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = 0.$$

Amongst other minor theorems here given I may mention the proposition that the kinetic energy imparted by given impulses to a material system under given constraints is a maximum. All the analysis is so elegant that Sir William Rowan Hamilton said the work could only be described as a scientific poem. It may be interesting to note that Lagrange remarked that mechanics was really a branch of pure mathematics analogous to a geometry of four dimensions, namely the time and the three coordinates of the point in space. At first no printer could be found who would publish the book; but Legendre at last persuaded a Paris firm to undertake it, and it was issued under his supervision in 1788.

In 1787 Frederick died, and Lagrange who had found the climate of Berlin very trying gladly accepted the offer of Louis XVI. to migrate to Paris. He received similar invitations from Spain and Naples. In France he was received with every mark of distinction, and special apartments in the Louvre were prepared for his reception. For the first two years of his residence here he was seized with an attack of melancholy, and even the printed copy of his *Mechanics* on which he had worked for a quarter of a century lay for more than two years unopened on his desk. Curiosity as to the results of the French revolution first stirred him out of his lethargy, a curiosity which soon turned to alarm as the revolution developed. It was about the same time, 1792, that the

unaccountable sadness of his life and his timidity moved the compassion of a young girl who insisted on marrying him, and proved a devoted wife, to whom he became warmly attached. Although the decree of October 1793, which ordered all foreigners to leave France, specially exempted him by name, he was preparing to escape when he was offered the presidency of the commission for the reform of weights and measures. The choice of the units finally selected was largely due to him, and it was mainly owing to his influence that the decimal subdivision was accepted by the commission of 1799. The general idea of the decimal system was taken from a work by Thomas Williams entitled *Method...for fixing an universal standard for weights and measures*, published in London in 1788. This almost unknown writer has hardly received the credit due to his suggestion.

Though Lagrange had determined to escape from France while there was yet time, he was never in any danger; and the different revolutionary governments (and at a later time Napoleon) loaded him with honours and distinctions. A striking testimony to the respect in which he was held was shewn in 1796 when the French commissary in Italy was ordered to attend in full state on Lagrange's father, and tender the congratulations of the republic on the achievements of his son, who "had done honour to all mankind by his genius, and whom it was the special glory of Piedmont to have produced."

In 1795 Lagrange was appointed to a mathematical chair at the newly established École normale which only enjoyed a brief existence of four months. His lectures here were quite elementary and contain nothing of any special importance, but they were published because the professors had to "pledge themselves to the representatives of the people and to each other neither to read nor to repeat from memory," and the discourses were ordered to be taken down in shorthand in order to enable the deputies to see how the professors acquitted themselves. His additions to Euler's *Algebra* were written about this time.

On the establishment of the École polytechnique in 1797

Lagrange was made a professor; and his lectures there are described by mathematicians who had the good fortune to be able to attend them, as almost perfect both in form and matter. Beginning with the merest elements he led his hearers on until almost unknown to themselves they were themselves extending the bounds of the subject: above all he impressed on his pupils the advantage of always using general methods expressed in a symmetrical notation. His lectures on the differential calculus form the basis of his *Théorie des fonctions analytiques* which was published in 1797. This work is the extension of an idea contained in a paper he had sent to the Berlin Memoirs in 1772, and its object is to substitute for the differential calculus a group of theorems based on the development of algebraic functions in series. A somewhat similar method had been previously used by Landen in his *Residual analysis*, published in London in 1764 (see p. 363). Lagrange believed that he could thus get rid of those difficulties connected with the use of infinitely large or infinitely small quantities which philosophers professed to see in the usual treatment of the differential calculus. The book is divided into three parts; of these the first treats of the general theory of functions, and gives an algebraic proof of Taylor's Theorem, the validity of which is however open to question; the second deals with the applications to geometry; and the third with the applications to mechanics. Another treatise on the same lines was his *Leçons sur le calcul des fonctions*, issued in 1805. These works may be considered as the starting-point for the researches of Cauchy and Jacobi. At a later period Lagrange reverted to the use of infinitesimals in preference to founding the differential calculus on a study of algebraic forms: and in the preface to the second edition of the *Mécanique*, which was issued in 1811, he justifies their use and concludes by saying that "when we have grasped the spirit of the infinitesimal method, and have verified the exactness of its results either by the geometrical method of prime and ultimate ratios or by the analytical method of derived functions, we may

double of which he calls the *vis viva*, as the measure of a force; according as the force is "passive" or "active."

The series quoted by Leibnitz comprise those for e^x, $\log(1+x)$, $\sin x$, $\operatorname{vers} x$, and $\tan^{-1}x$. All of these had been previously published, and he rarely if ever added any demonstrations. In 1693 he explained the method of expansion by indeterminate coefficients, though his applications were not free from error.

To sum the matter up briefly, it seems to me that Leibnitz's work exhibits great skill in analysis; but wherever he leaves his symbols and attempts to interpret his results or deal with concrete cases he commits blunders; and on the whole I think his mathematical work is overrated. No doubt the demands of politics and philosophy on his time may have prevented him from elaborating any subject completely or writing any systematic exposition of his views; but they are no excuse for the mistakes of principle which occur so frequently in his papers. Some of his memoirs contain suggestions of methods which have now become valuable means of analysis, such as the use of determinants and indeterminate coefficients. But when a writer of manifold interests like Leibnitz throws out innumerable suggestions, some of them are likely to turn out valuable; and to enumerate these (which he never worked out) without reckoning the others which are wrong seems to me to give a wholly false impression of the value of his work.

Leibnitz was only one amongst several continental writers whose papers in the *Acta eruditorum* familiarized mathematicians with the use of the differential calculus. The most important of these were Jacob and John Bernouilli, both of whom were warm friends and admirers of Leibnitz, and to whose unselfish devotion his reputation is largely due. Not only did they take a prominent part in nearly every mathematical question then discussed, but nearly all the leading mathematicians on the continent for the first half of the eighteenth century came directly or indirectly under the influence of the teaching of one or both of them.

employ infinitely small quantities as a sure and valuable means of shortening and simplifying our proofs."

His *Résolution des équations numériques* published in 1798 was also the fruit of his lectures at the Polytechnic. In this he gives the method of approximating to the real roots of an equation by means of continued fractions and enunciates several other theorems. In a note at the end he shows how Fermat's theorem that $a^{p-1} - 1 \equiv 0 \pmod p$, where p is a prime and a is prime to p, combined with a certain suggestion due to Gauss may be applied to give the complete algebraical solution of any binomial equation. He also here explained how the equation whose roots are the squares of the differences of the roots of the original equation may be used so as to give considerable information as to the position and nature of those roots.

The theory of the planetary motions had formed the subject of some of the most remarkable of Lagrange's Berlin papers. In 1806 the subject was reopened by Poisson who in a paper read before the French Academy shewed that Lagrange's formulæ led to certain limits for the stability of the orbits. Lagrange, who was present, now discussed the whole subject afresh, and in a memoir communicated to the Academy in 1808 explained how by the variation of arbitrary constants the periodical and secular inequalities of any system of mutually interacting bodies could be determined.

In 1810 Lagrange commenced a thorough revision of the *Mécanique analytique*, but he was only able to complete about two-thirds of it before his death.

In appearance he was of medium height, and slightly formed, with pale blue eyes, and a colourless complexion. In character he was nervous and timid, he detested controversy, and to avoid it willingly allowed others to take the credit for what he had himself done.

Lagrange was above all a student of pure mathematics: he sought and obtained abstract results of great generality and was content to leave the applications to others. Indeed no inconsiderable part of the discoveries of his great contemporary

Laplace consists of the application of the Lagrangian formulae to the facts of nature; for example his papers on the velocity of sound and the secular acceleration of the moon are all implicitly included in Lagrange's results. The only difficulty in reading Lagrange is that of the subject matter and the extreme generality of his processes; but his analysis is "as lucid and luminous as it is symmetrical and ingenious." A recent writer, speaking of Lagrange, says truly that he took a prominent part in the advancement of almost every branch of pure mathematics. Like Diophantus and Fermat he possessed a special genius for the theory of numbers, and in this subject he gave solutions of most of the problems which had been proposed by Fermat, and added some theorems of his own. He created the calculus of variations. To him too the theory of differential equations is indebted for its position as a science rather than a collection of ingenious artifices for the solution of particular problems. To the calculus of finite differences he contributed the formula of interpolation which bears his name. But above all he impressed on mechanics (which it will be remembered he considered a part of pure mathematics) that generality and completeness towards which his labours invariably tended. His works have been edited by M. Serret and published by the French government in 7 vols. Paris, 1867–1877. Delambre's account of his life is printed in the first volume.

Pierre Simon Laplace was born at Beaumont-en-Auge in Normandy on March 23, 1749 and died at Paris on March 5, 1827. He was the son of a small cottager or perhaps a farm labourer, and owed his education to the interest his abilities and engaging presence excited in some wealthy neighbours. Very little is known of his early years; for when he became distinguished he held himself aloof both from his relatives and from those who had assisted him. A similar pettiness of character marked many of his actions. It would seem that from a pupil he became an usher in the school at Beaumont; but having procured a letter of introduction to d'Alembert

he went to Paris to push his fortune. A paper on the principles of mechanics excited d'Alembert's interest, and on his recommendation a place in the military school was offered to Laplace.

Secure of a competency, Laplace now threw himself into original research, and in the next fifteen years, 1771—1785, he produced much of his original work in astronomy. This commenced with a memoir read before the French Academy in 1773 in which he showed that the planetary motions were stable, and carried the proof as far as the cubes of the eccentricities and inclinations. This was followed by several papers on points in the integral calculus, finite differences, differential equations, and astronomy. His astronomical researches are summed up in his work on the planets published in 1784; this also contains numerous theorems on the theory of attractions, of which some are taken from Legendre, but many are original.

During the next three or four years he produced some memoirs of exceptional power. Prominent among these are one read in 1784, and reprinted in the third volume of the *Mécanique céleste*, in which he completely determined the attraction of a spheroid on a particle outside it, and another on planetary inequalities which was presented in three sections in 1784, 1785, and 1786. The former of these, namely that on the attraction of a spheroid, is memorable for the introduction into analysis of the potential and of spherical harmonics or Laplace's coefficients. The potential of a body at any point is the sum of the mass of every element of the body when divided by its distance from the point. The name was first given by Green in 1828. Laplace showed that if the potential of a body at an external point was known, the attraction in any direction could be at once found. He also showed that the potential always satisfied the differential equation

$$\nabla^2 V \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

alluded to were presented to the French Academy and they are printed in the *Mémoirs présentés par divers savans.*

Laplace now set himself the task to write a work which should "offer a complete solution of the great mechanical problem presented by the solar system, and bring theory to coincide so closely with observation that empirical equations should no longer find a place in astronomical tables." The result is embodied in the *Exposition du système du monde* and the *Mécanique céleste.*

The former was published in 1796, and gives a general explanation of the phenomena with a summary of the history of astronomy, but omits all details. The nebular hypothesis was here first enunciated. According to this hypothesis the solar system has been evolved from a globular mass of incandescent gas rotating round an axis through its centre of mass. As it cooled, this mass contracted and successive rings broke off from its outer edge. These rings in their turn cooled, and finally condensed into the planets, while the sun represents the central core which is still left. A popular account of the theory, with certain corrections required by modern science, has been recently given by Sir William Thomson in the Proceedings of the Royal Institution for 1887. The arguments against the hypothesis are summed up in Faye's *Origine du monde.* According to the law published by Bode in 1778 the distances of the planets from the sun are nearly in the ratio of the numbers $0+4$, $3+4$, $6+4$, $12+4$, &c., the $(n+2)$th term being $2^n \times 3+4$. It would be an interesting fact if this could be deduced from Laplace's hypothesis, but so far as I am aware only one serious attempt to do so has been made, and the conclusion was that the law was not sufficiently exact to be worth more than a convenient means of remembering the general result. The substance of Laplace's hypothesis had been published by Kant in 1755 in his *Allgemeine Naturgeschichte* but it is doubtful whether Laplace was aware of this. The historical summary has always been esteemed one of the master-pieces of French literature, though it is not

altogether reliable for the later periods of which it treats, and it procured for the author the honour of admission to the forty of the French Academy.

The full analytical discussion of the solar system is given in the *Mécanique céleste* published in five vols.: vols. I. and II. in 1799; vol. III. in 1802; vol. IV. in 1805; and vol. V. in 1825. The first two volumes contain methods for calculating the motions of the planets, determining their figures, and resolving tidal problems. The third and fourth volumes contain the application of these formulæ, and also several astronomical tables. The fifth volume is mainly historical, but it gives as appendices the results of Laplace's latest researches. It is regrettable to have to add that theorems and formulæ are appropriated from numerous writers with but scanty acknowledgment, and the conclusions—which have been described as the organized result of a century of patient toil—are generally mentioned as if they were due to Laplace. It is said (for I have not looked into the matter myself) that the praise which he lavishes on Newton and Clairaut is only the cloak under which he appropriates the work of other and less known writers. The *Mécanique céleste* is by no means easy reading. Biot, who assisted Laplace in revising it for the press, says that Laplace himself was frequently unable to recover the details in the chain of reasoning, and if satisfied that the conclusions were correct he was content to insert the constantly recurring formula "Il est aisé à voir." The best tribute to the excellency of the work is that it left very little for his successors to add. It is not only the translation of the *Principia* into the language of the differential calculus, but it also completes those parts of which Newton had been unable to fill in the details.

Laplace went in state to beg Napoleon to accept a copy of his work and the following account of the interview is well authenticated, and so characteristic of all the parties concerned that I quote it in full. Some one had told Napoleon that the book contained no mention of the name of God; Napoleon

who was fond of putting embarrassing questions received it with the remark, "M. Laplace, they tell me you have written this large book on the system of the universe, and have never even mentioned its Creator." Laplace, who, though the most supple of politicians, was as stiff as a martyr on every point of his philosophy, drew himself up and answered bluntly, "Je n'avais pas besoin de cette hypothèse-là." Napoleon, greatly amused, told this reply to Lagrange, who exclaimed, 'Ah! c'est une belle hypothèse; ça explique beaucoup de choses.'

The remaining work by Laplace is his *Théorie analytique des probabilités* issued in 1812. The theory is stated to be only common sense expressed in mathematical language. The method of estimating the ratio of the number of favourable cases to the whole number of possible cases had been indicated by Laplace in a paper written in 1779. It consists in treating the successive values of any function as the coefficients in the expansion of another function with reference to a different variable. The latter is therefore called the generating function of the former. Laplace then shows how by means of interpolation these coefficients may be determined from the generating function. Next he attacks the converse problem, and from the coefficients he finds the generating function; this is effected by the solution of an equation in finite differences. The method is cumbersome, and in consequence of the increased power of analysis is now rarely used. An admirable summary of Laplace's reasoning is given by de Morgan in his article on Probability in the *Encyclopaedia Metropolitana*.

This treatise contains the greatest analytical achievement of Laplace, and one which is especially characteristic of his work. The method of least squares for the combination of numerous observations had been empirically given by Gauss and Legendre, but in the fourth chapter of this work Laplace gave a formal proof of it, on which the whole of the theory of errors has since been based. This was only effected by a most intricate analysis specially invented for the purpose, but the form in which it is presented is so meagre and unsatisfactory

that in spite of the uniform accuracy of his results it was at one time questioned whether he had actually gone through the difficult work he so briefly and often incorrectly indicates.

Amongst the minor discoveries of Laplace in pure mathematics I may mention his discussion (simultaneously with Vandermonde) of the general theory of determinants in 1772; his proof that every equation of an even degree must have at least one real quadratic factor, his reduction of the solution of linear differential equations to definite integrals, and his solution of the linear partial differential equation of the second order. He was also the first to consider the difficult problems involved in equations of mixed differences, and to prove that the solution of an equation in finite differences of the first degree and the second order might always be obtained in the form of a continued fraction. Besides these original discoveries he determined in his theory of probabilities the values of a large number of the more common definite integrals; and in the same book gave the general proof of the theorem enunciated by Lagrange for the development of any implicit function in a series by means of differential coefficients.

In 1819 Laplace published a popular account of his work on probability. This book bears the same relation to the *Théorie des probabilités* that the *Système du monde* does to the *Mécanique céleste*.

Laplace seems to have regarded analysis merely as a means of attacking physical problems, though the ability with which he invented the necessary analysis is almost phenomenal. As long as his results were true he took very little trouble to explain the steps by which he arrived at them; he never studied elegance or symmetry in his processes, and it was sufficient for him if he could by any means solve the particular question he was discussing. In these respects he stands in marked contrast to his great contemporary Lagrange. In theoretical physics the theory of capillary attraction is due to Laplace who accepted the idea propounded by Hawksbee in the *Phil. Trans.* 1709 that the phenomenon was due to a force of attraction

which was insensible at sensible distances. The part which deals with the action of a solid on a liquid and the mutual action of two liquids was not thoroughly worked out, but was ultimately completed by Gauss. Neumann later filled in a few details. Sir William Thomson has recently shown that if we assume the molecular constitution of matter, all the laws of capillary attraction can be deduced from the Newtonian law of gravitation (*Trans. Roy. Soc. Edinb.* 1862). Laplace in 1816 was the first to point out why Newton's theory of vibratory motion gave an incorrect value for the velocity of sound. The actual velocity is greater than that calculated by Newton in consequence of the heat developed by the sudden compression of the air which increases the elasticity and therefore the velocity of the sound transmitted. His only investigations in practical physics were those which were carried on by him jointly with Lavoisier in the years 1782 to 1784 on the specific heat of various bodies.

It would have been well for Laplace's reputation if he had been content with his scientific work, but above all things he coveted a decoration. The skill and rapidity with which he managed to change his politics as occasion required would be amusing if they had not been so servile. As Napoleon's power increased Laplace abandoned his republican principles (which had themselves gone through numerous changes, since they had faithfully reflected the opinions of the party in power) and begged the first consul to give him the post of minister of the interior. Napoleon who desired the support of men of science accepted the offer; but a little less than six weeks saw the close of Laplace's political career. Napoleon's memorandum on the subject is as follows. "Géomètre de premier rang, Laplace ne tarda pas à se montrer administrateur plus que médiocre; dès son premier travail nous reconnûmes que nous nous étions trompé. Laplace ne saisissait aucune question sous son véritable point de vue: il cherchait des subtilités partout, n'avait que des idées problématiques, et portait enfin l'esprit des 'infiniment petits' jusque dans l'administration."

Although expelled from office it was desirable to retain Laplace's allegiance. He was accordingly raised to the senate, and to the third volume of the *Mécanique céleste* he prefixed a note that of all the truths therein contained the most precious to the author was the declaration he thus made of his devotion towards the peace-maker of Europe. In copies sold after the restoration this was struck out. In 1814 it was evident that the empire was falling; Laplace hastened to tender his services to the Bourbons, and on the restoration was rewarded with the title of marquis. The contempt that his more honest colleagues felt for his conduct in the matter may be read in the pages of Paul Louis Courier, but the pettiness of his character must not make us forget how great were his services to science; and if it is any palliation of his conduct to the benefactors of his youth and his political friends it may be added that he never concealed his views on religion, philosophy, or mathematics however distasteful they might be to the authorities in power. His knowledge was very useful on the numerous scientific commissions on which he served, and probably accounts for the manner in which his political insincerity was overlooked.

That Laplace was vain and selfish is not denied by his warmest admirers; while his appropriation of the results of those who were comparatively unknown seems to be well established and is absolutely indefensible. Two of those whom he thus treated subsequently rose to distinction—Legendre in France and Young in England—and never forgave the injustice of which they had been the victims. It should however be stated that towards the close of his life and especially to the work of his pupils Laplace was both generous and appreciative, and in one case suppressed a paper of his own in order that a pupil might have the sole credit of the discovery.

His works were published in 7 volumes by the French government in 1843—7; and a new edition with considerable additional matter is now (1888) being issued.

Adrien Marie **Legendre** was born at Toulouse on Sept. 18,

1752 and died at Paris on Jan. 10, 1833. The leading events of his life are very simple and may be summed up briefly. He was educated at the Collège Mazarin in Paris, appointed professor at the military school in Paris in 1777, was a member of the Anglo-French commission of 1787 to connect Greenwich and Paris geodetically; served on several of the public commissions from 1792 to 1810; was made a professor at the Normal school in 1795; and subsequently held a few minor government appointments. The influence of Laplace was steadily exerted against his obtaining any office or public recognition, and Legendre who was a timid student accepted the obscurity to which the hostility of his colleague condemned him.

Legendre's analysis is of a high order of excellence and is second only to that produced by Lagrange and Laplace, though it is not nearly so original. His chief works are his *Geometry*, his *Theory of numbers*, his *Integral calculus*, and his *Elliptic functions*. These include the results of his various papers on these subjects. Besides these he wrote a treatise which gave the rule for the method of least squares, and two groups of memoirs, one on the theory of attractions, and the other on geodetical operations.

The memoirs on attractions are analyzed and discussed in chapters 20 to 25 of Todhunter's *History of Attraction*. The earliest of these, presented in 1783, was on the attraction of spheroids and marks the first distinct advance on the results of Maclaurin. This contains the introduction of Legendre's coefficients which are sometimes also called circular (or zonal) harmonics, and which are particular cases of Laplace's coefficients (see p. 385). It also includes the solution of a problem in which the potential is used, but this seems to have been due to a suggestion of Laplace, and its invention is properly attributed to him. The second memoir was communicated in 1784, and is on the form of equilibrium of a mass of rotating liquid which is approximately spherical. The third written in 1786 is on the attraction of confocal ellipsoids. The fourth is on the figure which a fluid planet would assume, and its law of density.

His papers on geodesy are three in number and were presented to the Academy in 1787 and 1788. The most important result is that by which a spherical triangle may be treated as plane, provided certain corrections are applied to the angles. In connection with this subject he paid considerable attention to geodesics.

The method of least squares was enunciated in his *Nouvelles méthodes* published in 1806, to which supplements were added in 1810 and 1820. Gauss independently had arrived at the same result, had used it in 1795, and published it and the law of facility in 1809. Laplace was the earliest writer to give a proof of it: this was in 1812 (see p. 388).

Of the books produced by Legendre the one most widely known is his *Éléments de géométrie* which was published in 1794, and was generally adopted on the continent as a substitute for Euclid. The later editions contain the elements of trigonometry, and the proofs of the irrationality of π and π^2 (see p. 371). An appendix on the difficult question of the theory of parallel lines was issued in 1803, and is bound up with most of the subsequent editions.

His *Théorie des nombres* was published in 1798, and appendices were added in 1816 and 1825; the third edition issued in two volumes in 1830 includes the results of his various later papers, and still remains a standard work on the subject. It may be said that he here carried the subject as far as was possible by the application of ordinary algebra; but he did not realize that it might be regarded as a higher arithmetic and so form a distinct subject in mathematics.

The law of quadratic reciprocity, which connects any two odd primes and which Gauss called "the gem of arithmetic," is first proved in this book, but the result had been enunciated in a memoir of 1785. The theorem is as follows. If p be a prime and n be prime to p then we know that the remainder when $n^{\frac{1}{2}(p-1)}$ is divided by p is either $+1$ or -1. Legendre denoted this remainder by $\left(\frac{n}{p}\right)$. When the remainder is $+1$ it is

possible to find a square number which when divided by p leaves a remainder n, in other words n is a quadratic residue of p: when the remainder is -1 there exists no such square number, and n is a non-residue of p. The law of quadratic reciprocity is expressed by the theorem that if a and b are any odd primes then

$$\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = (-1)^{\frac{1}{4}(a-1)(b-1)};$$

thus if b is a residue of a then a is also a residue of b, unless both of the primes a and b are of the form $4m+3$.

In other words, if a and b be odd primes we know that

$$a^{\frac{1}{2}(b-1)} \equiv \pm 1 \pmod b, \text{ and } b^{\frac{1}{2}(a-1)} \equiv \pm 1 \pmod a;$$

but by Legendre's law the two ambiguities will either be both positive or both negative, unless a and b are both of the form $4m+3$. Thus if one odd prime is a non-residue of another then the latter will be a non-residue of the former. Gauss and Kummer have subsequently proved similar laws of cubic and biquadratic reciprocity; and an important branch of the theory of numbers has been based on these researches. This work also contains the useful theorem by which, when it is possible, an indeterminate equation of the second degree can be reduced to the form $ax^2+by^2+cz^2=0$, and a discussion of numbers which can be expressed as the sum of three squares.

The *Exercises de calcul intégral* was published in three volumes 1811, 1817, 1826. Of these the third and most of the first are devoted to elliptic functions: the bulk of this being ultimately included in the *Fonctions elliptiques*. The contents of the remainder of the treatise are of a very miscellaneous character: they include integration by series, definite integrals, and in particular an elaborate discussion of the Beta and the Gamma functions.

The *Traité des fonctions elliptiques* was issued in two volumes in 1825 and 1826 and is the most important of Legendre's works.

A third volume containing three memoirs on the researches of Abel and Jacobi was added a few weeks before his death. Legendre's investigations had commenced with a paper written in 1786 on elliptic arcs; and he subsequently wrote numerous other papers on the subject, which he treated entirely as a branch of the integral calculus. Tables of the elliptic integrals were constructed by him. The modern treatment of the subject is founded on that of Abel and Jacobi. The superiority of their methods was at once recognized by Legendre, and almost the last act of his life was to recommend those discoveries which he knew would consign his own labours to oblivion.

This may serve to remind us of a fact which I wish to specially emphasize, namely, that Gauss, Abel, Jacobi, and some others of the mathematicians alluded to in the next chapter were contemporaries of the members of the French school.

The creation of modern geometry.

While Euler, Lagrange, Laplace, and Legendre were perfecting analysis another group of French mathematicians were developing geometry by methods similar to those previously used by Desargues and Pascal. The most eminent of those who created what is called modern geometry are Monge, Poncelet, and Carnot; and its development in more recent times is largely due to Steiner, Von Staudt, and Cremona.

Gaspard **Monge*** was born at Beaune on May 10, 1746 and died at Paris on July 28, 1818. He was the son of a small pedlar, and was educated in the schools of the Oratorians, in one of which he subsequently became an usher. A plan which he had made of some neighbouring village fell into the hands of an officer who recommended the military authorities to admit him to their training-school at Mézières. His birth precluded his receiving a commission in the army, but his attendance at an annexe of the school where surveying and drawing were taught was tolerated, though he was told that he was not

* See *Essai historique sur les travaux...de Monge*, by Dupin, Paris, 1819.

sufficiently well born to be allowed to attempt problems which required calculation. At last his opportunity came. A plan of a fortress having to be drawn from the data supplied by certain observations, he did it by a geometrical construction. At first the officer in charge refused to receive it, because etiquette required that not less than a certain time should be used over making such drawings, but the superiority of the method over that then taught was so obvious that it was accepted; and in 1768 Monge was made professor, on the understanding that the results of his descriptive geometry were to be a military secret confined to officers of certain ranks.

In 1780 he was appointed to a chair of mathematics in Paris, and this with several provincial appointments which he held gave him a comfortable income. The earliest paper of any special importance which he communicated to the French Academy was one in 1781 in which he discussed the lines of curvature drawn on a surface. These had been first considered by Euler in 1760, and defined as those normal sections whose curvature was a maximum or a minimum. Monge treated them as the locus of those points on the surface at which successive normals intersect, and thus obtained the general differential equation. He applied his results to the central quadrics in 1795. In 1786 he published his well-known work on statics.

Monge eagerly embraced the doctrines of the revolution. In 1792 he became minister of the marine, and assisted the committee of public safety in utilizing science for the defence of the republic. When the Terrorists obtained power he was denounced, and only escaped the guillotine by a hasty flight. On his return in 1794 he was made a professor at the short-lived Normal school where he gave lectures on descriptive geometry; the notes of these were published under the regulation above alluded to (see p. 380). In 1796 he went to Italy on the roving commission which was sent with orders to compel the various Italian towns to offer any pictures, sculpture, or other works of art that they might possess as a present or in

tion of contributions to the French republic for removal to Paris. In 1798 he accepted a mission to Rome, and after executing it joined Napoleon in Egypt. Thence after the naval and military victories of England he escaped to France. He was then made professor at the Polytechnic school, where he gave lectures on descriptive geometry; these were published in 1800 in the form of a text-book entitled *Géométrie descriptive*. This work contains propositions on the form and relative position of geometrical figures deduced by the use of transversals and the theory of perspective: the latter includes the art of representing in two dimensions geometrical objects which are of three dimensions; a problem which Monge usually solved by the aid of two diagrams, one being the plan and the other the elevation. Monge also discussed the question as to whether if in solving a problem certain subsidiary quantities introduced to facilitate the solution become imaginary the validity of the solution is thereby impaired, and he shewed that the result would not be affected. On the restoration he was deprived of all his offices and honours, an insult which preyed on his mind and which he did not long survive.

Most of his miscellaneous papers are embodied in his works *Application de l'algèbre à la géométrie* published in 1805, and *Application de l'analyse à la géométrie*, the fourth edition of which, published in 1819, was revised by him just before his death. It contains among other results his solution of a partial differential equation of the second order.

Lazare Nicholas Marguerite **Carnot**, born at Nolay on May 13, 1753 and died at Magdeburg on Aug. 22, 1823, was educated in Burgundy, and obtained a commission in the engineer corps of Condé. Although in the army, he continued his mathematical studies in which he felt great interest. His first work, published in 1784, was on machines; it contains a statement which foreshadows the principle of energy as applied to a falling weight, and the earliest proof of the fact that kinetic energy is lost in the collision of bodies. On the outbreak of the revolution in 1789 he threw himself into

politics. In 1793 he was elected on the committee of public safety, and the victories of the French army were largely due to his powers of organization and enforcing discipline. He continued to occupy a prominent place in every successive form of government till 1796, when having opposed Napoleon's *coup d'état* he had to fly from France. He took refuge in Geneva, and there in 1797 issued his *Metaphysics of the infinitesimal calculus*. In 1802 he assisted Napoleon, but his sincere republican convictions were inconsistent with the retention of office. In 1803 he produced his *Geometry of position and an essay on transversals*. This work deals with projective rather than descriptive geometry, it also contains an elaborate discussion of the geometrical meaning of negative roots of an algebraical equation. In 1814 he offered his services to fight for France, though not for the empire; and on the restoration he was exiled.

Jean Victor **Poncelet**, born at Metz on July 1, 1788 and died at Paris on Dec. 22, 1867, held a commission in the French engineers. Having been made a prisoner in the French retreat from Moscow in 1812 he occupied his enforced leisure by writing the *Traité des propriétés projectives des figures*, which was published in 1822, and has since been one of the best known text-books on modern geometry. By means of projection, reciprocation, and homologous figures he established all the chief properties of conics and quadrics. His treatise on practical mechanics in 1826, his memoir on water-mills in 1826, and his report on the English machinery and tools exhibited at the International exhibition held in London in 1851, deserve mention. He contributed numerous articles to Crelle's journal. The most valuable of these deal with the explanation of imaginary solutions in geometrical problems by the aid of the doctrine of continuity.

The development of mathematical physics.

It will be noticed that Lagrange, Laplace, and Legendre mostly occupied themselves with analysis, geometry, and astronomy. I am inclined to regard Cauchy and the French mathematicians of the present day as belonging to a different school of thought to that considered in this chapter and I place them amongst modern mathematicians, but I think that Fourier, Poisson, and the majority of their contemporaries are the lineal successors of Lagrange and Laplace. If this view be correct, it would seem that the later members of the French school devoted themselves mainly to the application of mathematical analysis to physics. Before considering these mathematicians I may mention the great English experimental physicists who were their contemporaries, and whose merits have only recently received an adequate recognition. Chief among these are Cavendish and Young.

The honourable *Henry* **Cavendish** was born at Nice on Oct. 10, 1731 and died in London on Feb. 24, 1810. He created experimental electricity and has a better claim than either Priestley or Lavoisier to be described as the founder of exact chemistry. I mention him here on account of his experiments in 1798 to determine the density of the earth, by estimating its attraction as compared with that of two given lead balls: the result is that the mean density of the earth is about five and a half times that of water.

Sir Benjamin Thomson, **Count Rumford**, born at Concord on March 26, 1753 and died at Auteuil on Aug. 21, 1815, was of English descent and fought on the side of the loyalists in the American War of secession: on the conclusion of peace, he settled in England, but subsequently entered the service of Bavaria where his military and civil powers of organization proved of great value. At a later period he again resided in England, and when there founded the Royal Institution. The majority of his papers were communicated to the Royal Society

of London; of these the most important is his memoir in which he shewed that heat and work are mutually convertible.

Thomas Young, born at Milverton on June 13, 1773 and died in London on May 10, 1829, was among the most eminent physicists of his time. He seems as a boy to have been somewhat of a prodigy, being well read in modern languages and literature as well as in science; he always kept up his literary tastes and it was he who first furnished the key to decipher the Egyptian hieroglyphics. He was destined to be a doctor, and after attending lectures at Edinburgh and Göttingen entered at Emmanuel College, Cambridge, from which he took his degree in 1799; and to his stay at the university he attributed much of his future distinction. His medical career was not particularly successful, and his favorite maxim that a medical diagnosis is only a balance of probabilities was not appreciated by his patients who looked for certainty in return for their fee. Fortunately his private means were ample. Several papers contributed to various learned societies from 1798 onwards prove him to have been a mathematician of considerable power; but the researches which have immortalized his name are those by which he laid down the laws of interference of waves and of light, and was thus able to overcome the chief difficulties in the way of the acceptance of the undulatory theory of light. For further details see his life and works by G. Peacock, 4 vols. 1855.

Another experimental physicist of the same time and school was *William Hyde* Wollaston, who was born at Dereham on Aug. 6, 1766 and died in London on Dec. 22, 1828. He was educated at Caius College, Cambridge, of which society he was a fellow. Besides researches on experimental optics, he is celebrated for the improvements he effected in astronomical instruments.

Another well-known writer of the same period was *John* Dalton who was born in Cumberland on Sept. 5, 1766 and died at Manchester on July 27, 1844. Dalton determined the laws of the expansions of gases, the tension of vapours, and the

specific heats of gases. He founded the atomic theory in chemistry.

It will be gathered from these notes that the English school of physicists at the beginning of this century were mostly concerned with the experimental side of the subject. But in fact no satisfactory theory could be formed without some similar careful determination of the facts. The most eminent French physicists of the same time were Fourier, Poisson, Ampère, and Fresnel. Their method of treating the subject is more mathematical than that of their English contemporaries, and the two first named were distinguished for general mathematical ability.

Jean Baptiste Joseph **Fourier** was born at Auxerre on March 21, 1768 and died at Paris on May 16, 1830. He was the son of a tailor, and was educated by the Benedictines. The commissions in the scientific corps of the army were as is still the case in Russia reserved for those of good birth, and being thus ineligible he accepted a military lectureship on mathematics. He took a prominent part in his own district in promoting the Revolution, and was rewarded by an appointment in 1795 in the Normal school, and subsequently by a chair at the Polytechnic school.

He went with Napoleon on his eastern expedition in 1798, and was made governor of lower Egypt. Cut off from France by the English fleet he organized the workshops on which the French army had to rely for their munitions of war. He also contributed several mathematical papers to the Egyptian Institute which Napoleon founded at Cairo with a view of weakening English influence in the East. After the British victories and the capitulation of General Menou in 1801, he returned to France and was made prefect of Grenoble, and it was while there that he made his experiments on the propagation of heat. He moved to Paris in 1816. In 1822 he published his *Théorie analytique de la chaleur*, in which he bases his reasoning on Newton's law of cooling, namely that the

flow of heat between two adjacent molecules is proportional to the infinitely small difference of their temperatures. He states that the theory demands that the temperature of stellar space should be between $-50°$ C. and $-60°$ C., a conclusion which it has as yet been impossible to verify. In this work he shews that any function of a variable, whether continuous or discontinuous, can be expanded in a series of sines of multiples of the variable; a result which is constantly used in modern analysis. Lagrange had given particular cases of the theorem and had implied that the method was general, but he had not pursued the subject.

Fourier left an unfinished work on determinate equations which was edited by Navier, and published in 1831; this contains much original matter, in particular there is a demonstration of Fourier's theorem on the position of the roots of an algebraical equation. Lagrange had shown how the roots of an algebraical equation might be separated by means of another equation whose roots were the squares of the differences of the roots of the original equation. Budan in 1807 and 1811 had enunciated the theorem generally known by the name of Fourier, but the demonstration was clumsy and not altogether satisfactory. Fourier's proof is the same as that usually given in text-books on the theory of equations. The final solution of the problem was given in 1829 by Jacques Charles François Sturm (who was born in 1803 and died in 1855).

Among Fourier's contemporaries who were interested in the theory of heat the most eminent was *Sadi* Carnot, a son of the eminent geometrician mentioned above. Sadi Carnot was born at Paris in 1796 and died there of cholera in August, 1832; he was an officer in the French army. In 1824 he issued a short work entitled *Reflexions sur la puissance motrice du feu* in which he attempted to determine in what way heat produced its mechanical effect. He made the mistake of assuming that heat was material, but his essay was the commencement of the modern theory of thermodynamics.

Siméon Denis Poisson, born at Pithiviers on June 21, 1781 and died at Paris on April 25, 1840, is almost equally distinguished for his applications of mathematics to mechanics and to physics. His father had been a common soldier, and on his retirement was given some small administrative post in his native village: when the revolution broke out he appears to have assumed the government of the place, and being left undisturbed became a person of some local importance. The boy was put out to nurse, and he used to tell how one day his father coming to see him found that the nurse had gone out on pleasure bent, while she had left him suspended by a small cord to a nail fixed in the wall. This she explained was a necessary precaution to prevent him from perishing under the teeth of the various animals and animalculæ that roamed on the floor. Poisson used to add that his gymnastic efforts carried him incessantly from one side to the other, and it was thus in his tenderest infancy that he commenced those studies on the pendulum that were to occupy so large a part of his mature age.

He was educated by his father, and destined much against his will to be a doctor. His uncle offered to teach him the art; and began by making him prick the veins of cabbage leaves with a lancet. When perfect in this, he was allowed to put on blisters; but in almost the first case he did this by himself the patient died in a few hours, and though all the medical practitioners of the place assured him that "the event was a very common one" he vowed he would have nothing more to do with the profession. Returning home he found amongst the official papers sent to his father a copy of the questions set at the Polytechnic school, and at once found his career. At the age of seventeen, he entered the Polytechnic, and his abilities at once excited the interest of Lagrange and Laplace whose friendship he retained to the end of their lives. A memoir on finite differences which he wrote when only eighteen was reported on so favorably by Legendre that it was ordered to be published in the *Recueil des savants étran-*

gers. Directly he had finished his course he was made a lecturer at the school, and he continued through his life to hold various government scientific posts and professorships. He was somewhat of a socialist and remained a rigid republican till 1815 when, with a view to making another empire impossible, he became a legitimist. He took however no active part in politics and devoted all his spare time to mathematics.

His works and memoirs are between three and four hundred in number. The chief treatises which he wrote were his *Traité de mécanique**, 2 vols. 1811 and 1833, which was long a standard work; his *Théorie nouvelle de l'action capillaire*, 1831; his *Théorie mathématique de la chaleur*, 1835, to which a supplement was added in 1837; and his *Recherches sur la probabilité des jugements*, 1837. He had intended if he had lived to write a work which should cover all mathematical physics and in which these would have been incorporated.

Of his memoirs on the subject of pure mathematics the most important are those on definite integrals, and Fourier's series (these are to be found in the *Journal polytechnique* from 1813 to 1823, and in the *Mémoires de l'Académie* for 1823), their application to physical problems constituting one of his chief claims to distinction; his essay on the calculus of variations (*Mém. de l'Acad.*, 1833); and his papers on the probability of the mean results of observations (*Connaiss. des Temps*, 1827 &c.).

Perhaps the most original of his memoirs in applied mathematics are those on the theory of electricity and magnetism which created a new branch of mathematical physics. He supposed that the results were due to the attractions and repulsions of imponderable particles. The most important of

* Among Poisson's contemporaries who studied mechanics and of whose works he made use I must mention *Louis Poinsot* who was born in Paris on Jan. 3, 1777 and died there on Dec. 5, 1859. In his *Statique* published in 1803 he treated the subject without any explicit reference to dynamics: the theory of couples is almost wholly due to him (1806), as also the motion of a body in space under the action of no forces.

those on physical astronomy are the two entitled *Sur les inégalités séculaires des moyens mouvements des planètes*, and *Sur la variation des constantes arbitraires dans les questions de mécanique* (*Journal polytechnique*, 1809). In these Poisson discusses the question of the stability of the planetary orbits, which had already been settled by Lagrange to the first degree of approximation for the disturbing forces, and shews that the result can be extended up to the third order of small quantities. These were the memoirs which led to Lagrange's famous memoir of 1808. Poisson also published a paper *Sur la libration de la lune* (*Connaiss. des Temps*, 1821); and another *Sur la mouvement de la terre autour de son centre de gravité* (*Mém. de l'Acad.*, 1827). His two most important memoirs on the theory of attraction are *Sur l'attraction des sphéroides* (*Connaiss. des Temps*, 1829) and *Sur l'attraction d'un ellipsoide homogène* (*Mém. de l'Acad.*, 1835). The substitution of the correct equation for the potential $\nabla^2 V^2 = -4\pi\rho$ for Laplace's form of it $\nabla^2 V = 0$ was first published in the *Bulletin de la Société Philomatique*, 1813. Lastly I may mention his memoir on the theory of waves (*Mém. de l'Acad.*, 1825). A complete list of his works is given in vol. II. of Arago's works.

André Marie Ampère was born at Lyons on January 22, 1775 and died at Marseilles on June 10, 1836. He was widely read in all branches of learning, and lectured and wrote on many of them, but after the year 1809 when he was made professor of analysis at the Polytechnic school in Paris he confined himself almost entirely to mathematics and science. His papers on the connection between electricity and magnetism were written in 1820. According to his theory propounded in 1826 a molecule of matter which can be magnetized is traversed by a closed electric current, and magnetisation is produced by any cause which makes the direction of these currents in the different molecules of the body approach parallelism. For further details of his writings see Valson's *Étude sur la vie et les ouvrages d'Ampère*, Lyons, 1885.

Augustin Jean Fresnel, born at Broglie on May 10, 1788

and died at Ville-d'Avray on July 14, 1827, was a civil engineer by profession, but he devoted his leisure to the study of physical optics. The undulatory theory of light which Hooke, Huygens, and Euler had supported on *à priori* grounds had been based on experiment by the researches of Young. Fresnel deduced the mathematical consequences of these experiments, and explained the phenomena of interference both of ordinary and polarized light.

His friend and contemporary *Jean Baptiste* Biot who was born at Paris on April 21, 1774 and died there in 1862, requires a word or two in passing. Most of his mathematical work was in connection with the subject of optics and especially the polarization of light. His systematic works were all produced within the years 1805 and 1817: a selection of his more valuable memoirs was published in Paris in 1858.

François Jean Dominique Arago was born at Estagel in the Pyrenees on Feb. 26, 1786 and died in Paris on Oct. 2, 1853. He was educated at the Polytechnic school Paris, and we gather from his autobiography that however distinguished were the professors of that institution they were remarkably incapable of imparting their knowledge or maintaining discipline. In 1804 he was made secretary to the observatory, and from 1806 to 1809 he was engaged in measuring a meridian arc in order to determine the exact length of a metre. He was then made one of the astronomers at Paris, given a residence there, and made a professor at the Polytechnic school where he enjoyed a marked success as a lecturer. He subsequently gave popular lectures on astronomy which were both lucid and accurate, a combination of qualities which was rarer then than now.

He reorganized the national observatory, the management of which had long been inefficient, but in doing this he shewed himself dictatorial and passionate, and the same defects of character revealed themselves in many of the events of his life. His earliest physical researches were on the pressure of steam at different temperatures, and the velocity of sound, 1818 to

1822. His magnetic observations mostly took place from 1823 to 1830. He discovered what has been called rotatory magnetism, and the fact that most bodies could be magnetized; these discoveries were completed and explained by Faraday. He warmly supported Fresnel's optical theories, and the two philosophers conducted together those experiments on the polarization of light which led to the inference that the vibrations of the luminiferous ether were transverse to the direction of motion, and that polarization consisted in a resolution of rectilinear motion into components at right angles to each other. The subsequent invention of the polariscope and discovery of rotatory polarization are due to Arago. The general idea of the experimental determination of the velocity of light in the manner subsequently effected by Fizeau and Foucault was suggested by him in 1838, but his failing eyesight prevented his arranging the details or making the experiments. He remained to the end a consistent republican, and after the *coup d'état* of 1852 though half blind and dying he resigned his post as astronomer rather than take the oath of allegiance. It is to the credit of Napoleon that he gave directions that the old man should be in no way disturbed and should be left free to say and do what he liked.

His works, which include *éloges* on most of the leading mathematicians of the last five or six centuries, have been edited by M. J. A. Barral and published in 14 vols. 1856—57.

It will be noticed that some of the last members of the French school were alive at a comparatively recent date, but nearly all their mathematical work was done before the year 1830; in any case they are the direct successors of the French writers who flourished at the commencement of this century, and seem to have been quite out of touch with the great German mathematicians of the early part of it on whose researches the most recent work is based. They are thus placed here though their writings are in some cases of a later date than those of Gauss, Abel, Jacobi and other mathematicians of recent times.

The introduction of analysis into England.

It will be remembered that the English mathematicians of the beginning of this century still confined themselves in general to strictly Newtonian methods. Almost the only exception was Ivory to whom the celebrated theorem in attractions is due.

James Ivory was born in Dundee in 1765 and died at Douglastown on Sept. 21, 1845. After graduating at St Andrews he became the managing partner in a flax spinning company in Forfarshire, but continued to devote most of his leisure to mathematics. In 1804 he was made professor at the Royal Military College at Marlow, which is now moved to Sandhurst. He contributed numerous papers to the *Philosophical Transactions*, the most remarkable being those on attractions. In one of these, in 1809, he showed how the attraction of a homogeneous ellipsoid on an external point is a multiple of that of another ellipsoid on an internal point: the latter can be easily obtained. He criticized Laplace's solution of the method of least squares with unnecessary bitterness, but only proved his incompetence to understand Laplace.

The introduction of the notation of the differential calculus into England was due to three undergraduates at Cambridge, Babbage, Peacock, and Herschel, to whom a word or two may be devoted. The original stimulus came from French sources and I therefore place these remarks at the close of my account of the French school, but I should add that the English mathematicians of this century at once struck out a line quite independent of their French contemporaries.

Charles Babbage was born at Teignmouth on Dec. 26, 1792 and died in London on Oct. 18, 1871. He entered at Trinity College, Cambridge, in 1811, and in the next year joined Herschel and Peacock in founding the Analytical Society, which Babbage explained was to advocate "the principles of pure *d*-ism as opposed to the *dot*-age of the university." In 1816 the society published a translation of

Lacroix's differential calculus, which was followed in 1820 by two volumes of examples: all elementary works on this subject since published have abandoned the exclusive use of the fluxional notation. It should be noticed in passing that Lagrange and Laplace like Sir William Thomson and other modern writers use both the fluxional and the differential notation. It was the exclusive use of the former that was so hampering. Babbage will always be famous for his invention of an analytical machine which could not only perform the ordinary processes of arithmetic, but could tabulate the values of any function and print the results. The machine was never finished—somewhat owing to Babbage's own fault—but the drawings of it now deposited at Kensington satisfied a scientific commission that it could be constructed. Babbage was Lucasian professor at Cambridge from 1828 to 1839, though by an abuse which was then possible he never resided or lectured.

George Peacock, born at Denton on April 9, 1791 and died at Ely on Nov. 8, 1858, was educated at Trinity College, Cambridge, of which society he was a fellow and tutor. In 1837 he was appointed Lowndean professor, and in 1839 was made dean of Ely. His influence on the Cambridge and English mathematicians of his time was considerable, but he has left few remains except his *Examples illustrative of the use of the differential calculus*, 1820; his *Algebra*, 1830 and 1842; and his *Report on recent progress in analysis*, 1833, which commenced those valuable summaries of scientific progress which enrich many of the annual volumes of the British Association.

Sir John Frederick William Herschel was born at Slough on March 7, 1792 and died at Collingwood on May 11, 1871. He was educated at Eton and St John's College, Cambridge, and it was while an undergraduate there that he made the acquaintance of Babbage and Peacock. With youthful enthusiasm he proposed that they should enter into a compact to "do their best to leave the world wiser than they found it," and the introduction of the differential calculus into the university curriculum was proposed by his two friends as the first test of

their sincerity. His father was Sir William Herschel (1738–1822) who was the most illustrious astronomer of the last half of the last century, just as the son was amongst the most eminent astronomers of this century. Besides his numerous papers on astronomy, which lie outside the range of this book, his *Outlines of astronomy* published in 1849, his article on Light in the *Encyclopaedia Metropolitana*, and that on Meteorology in the 8th edition of the *Encyclopaedia Britannica* deserve mention for their mathematical or physical merits.

The work of this little society was supplemented by *Henry Parr* Hamilton, born at Edinburgh on April 3, 1794 and died at Salisbury on Feb. 7, 1880, who was also a fellow of Trinity College, Cambridge, and who in 1826 published an analytical geometry which was an improvement on anything then accessible to English readers. These text-books were soon replaced by better ones, but the latter lie outside the limits of this chapter.

CHAPTER XIX.

RECENT TIMES.

SECTION 1. *Elliptic and Abelian functions.*
SECTION 2. *The theory of numbers.*
SECTION 3. *Higher algebra.*
SECTION 4. *Modern geometry.*
SECTION 5. *Analytical geometry.*
SECTION 6. *Analysis.*
SECTION 7. *Astronomy.*
SECTION 8. *Mathematical physics.*

THE French school which flourished at the beginning of this century may be taken as ending with the death of Legendre and Poisson, at any rate as far as mathematicians of first-rate ability are concerned. The mathematicians of the nineteenth century have mostly specialized their work in one or more departments. They may roughly be divided into those who have specially studied pure mathematics (in which I should include theoretical dynamics and astronomy) and those who have specially studied physics: the latter subject requiring as it develops a fair knowledge of mathematics even on the part of those who treat it from the experimental side alone. Among the writers of this period I include Gauss, Abel, Cauchy, and a few others who, though they were contemporaries of the later years of Lagrange, Laplace, Legendre, and Poisson, stand quite apart from the French school, of which the latter mathematicians were the most distinguished members.

It is evidently impossible for me to discuss adequately the mathematicians of the age in which we live, especially as I purposely exclude from this work any detailed reference to living writers. I make therefore no attempt to give a complete history of this century, but as a sort of appendix to the preceding chapters I may specially mention the names of the

following mathematicians as among those who have contributed most powerfully to the recent progress of mathematics. I add the date of birth wherever I know it.

Niels Henrik *Abel*, born at Findoë on Aug. 5, 1802, and died at Arendal on April 6, 1829 (see p. 416): John Couch *Adams*, of St John's and Pembroke Colleges, Cambridge, born in Cornwall on June 5, 1819, and now Lowndean professor at the university of Cambridge: Paul Emile *Appell*, born at Strassburg 1855, and now professor in Paris: Siegfried Heinrich *Aronhold*, born at Angerburg on July 16, 1819: Sir Robert Stawell *Ball*, of Trinity College, Dublin, born at Dublin on July 1, 1840, and now astronomer royal of Ireland: Eugenio *Beltrami*, born at Cremona 1835, and now professor at Pavia: Joseph Louis François *Bertrand*, born at Paris in 1822, and now secretary of the French Academy: Friederich Wilhelm *Bessel*, born at Minden on July 22, 1784, and died at Königsberg on March 17, 1846 (see p. 438): Enrico *Betti*: Ludwig *Boltzmann*, professor of physics at the university of Vienna: George *Boole*, born at Lincoln on Nov. 2, 1815, and died at Cork on Dec. 8, 1864 (see p. 430): James *Booth*, born in county Leitrim on Aug. 25, 1806, and died in Buckinghamshire on April 15, 1878: Carl Wilhelm *Borchardt*, born at Berlin on Feb. 22, 1817: C. J. C. *Bouquet*: Francesco *Brioschi*: Charles *Briot*, born in 1817: Arthur *Cayley*, of Trinity College, Cambridge, born at Richmond in Surrey on Aug. 16, 1821, and now Sadlerian professor at the university of Cambridge: Augustin Louis *Cauchy*, born at Paris on Aug. 21, 1789, and died at Sceaux on May 25, 1857 (see p. 436): Michel *Chasles*, born at Epernon on Nov. 15, 1793, and died at Paris on Dec. 18, 1880 (see p. 433): Rudolph Julius Emmanuel *Clausius*, born at Cöslin on Jan. 2, 1822, and died at Bonn where he was professor of physics in August, 1888: Rudolph Frederick Alfred *Clebsch*: William Kingdon *Clifford*, born at Exeter on May 4, 1845, and died at Madeira on March 3, 1879 (see p. 434): Luigi *Cremona*: Morgan William *Crofton*: Jean Gaston *Darboux*, born at Nîmes in 1842, and now professor in Paris: George Howard

Darwin, of Trinity College, Cambridge, born in 1846, and now Plumian professor at the university of Cambridge: Julius Wilhelm Richard *Dedekind*, born at Brunswick on Oct. 6, 1831: Charles Eugène *Delaunay*, born at Lusigny on April 9, 1816, and drowned off Cherbourg on Aug. 5, 1872: Augustus *de Morgan*, born in Madras in June, 1806, and died in London on March 18, 1871 (see p. 430): Peter Gustav Lejeune *Dirichlet*, born at Düren on Feb. 13, 1805, and died at Göttingen on May 5, 1859 (see p. 421): Ferdinand Gotthold *Eisenstein*, born at Berlin on April 16, 1823, and died there on Oct. 11, 1852: Michael *Faraday*, born at Newington on Sept. 22, 1791, and died at Hampton Court on Aug. 25, 1867 (see p. 441): Jean Bernard Léon *Foucault*, born at Paris on Sept. 18, 1819, and died there on Feb. 11, 1868 (see p. 442): G. *Frobenius*: Lazarus *Fuchs*, born in Posen, 1833, now professor at Berlin: Evariste *Galois*, born at Paris on Oct. 26, 1811, and died there on May 30, 1832: Karl Friederich *Gauss*, born at Brunswick, April 23, 1777, and died at Göttingen on Feb. 23, 1855 (see p. 420): Adolph *Göpel*, born in September, 1812, and died in March, 1847: Paul *Gordan*: Hermann Günter *Grassmann*, born at Stettin on April 15, 1809; George *Green*, born near Nottingham in 1793, and died at Cambridge in 1841 (see p. 443): Georges Henri *Halphen*, born at Rouen, 1844, an officer in the French army: Sir William Rowan *Hamilton*, born in Dublin on Aug. 4, 1805, and died there on Sept. 2, 1865 (see p. 429): Peter Andreas *Hansen*, born in Schleswig on Dec. 8, 1795, and died at Gotha where he was head of the observatory on March 28, 1874: Hermann Ludwig Ferdinand von *Helmholtz*, born at Potsdam on Aug. 31, 1821, and now professor at the university of Berlin: Charles *Hermite*, born in Lorraine on Dec. 24, 1822, now professor in Paris: Ludwig Otto *Hesse*, born at Königsberg on April 22, 1811, and now professor at the university of Heidelberg: George William *Hill*, born in New York, 1838, and now in the office of the American Ephemeris (i.e. the nautical almanack), Washington: Karl Gustav Jacob *Jacobi*, born at Potsdam on Dec. 10, 1804, and died at Berlin on

and now professor in Paris: Henry John Stephen *Smith*, born in London on Nov. 2, 1826, and died at Oxford on Feb. 9, 1882 (see p. 424): Karl Georg Christian von *Staudt*, born at Rothenburg on Jan. 24, 1798, and died in 1867: Jacob *Steiner*, born at Utzensdorf on March 18, 1796: George Gabriel *Stokes*, of Pembroke College, Cambridge, born in Sligo on Aug. 13, 1819, and now Lucasian professor at the university of Cambridge: James Joseph *Sylvester*, of St John's College, Cambridge, born in London on Sept. 3, 1814, and now Savilian professor at the university of Oxford: Pafnutij *Tchebycheff*, born in Russia, 1821, and formerly professor at the university of St Petersburg: Sir William *Thomson*, of Peterhouse, Cambridge, born at Belfast in June, 1824, and now professor of natural philosophy at the university of Glasgow: Wilhelm Edward *Weber*, born at Wittenberg on Oct. 24, 1804: Karl *Weierstrass*, born at Ostenfelde on Oct. 31, 1815, and now professor at the university of Berlin: Gustav *Wiedemann*, born at Berlin on Oct. 2, 1826: Hieronymus Georg *Zeuthen*, born in Denmark, 1839, and now professor at the university of Copenhagen.

The above list is not and does not pretend to be exhaustive; and it is of only a few of those there mentioned that I here give even the briefest account. In spite of these limitations and of the considerable trouble I have taken over it I am not satisfied with this chapter; and though my friend Dr Glaisher has kindly read the proofs of part of it, added some names, and both he and Mr Forsyth have answered my numerous questions, yet the form in which I had cast it and the inherent difficulties of the subject have I fear prevented it from being otherwise than most imperfect. The quantity of matter produced during this century has in fact been so enormous* that no one can expect to do more than make himself acquainted with the work in some small department. I hope however

* For example, I infer from the catalogue of scientific papers which is periodically issued by the Royal Society, that something like 15,000 separate scientific memoirs are now published every year by the different societies and journals of Europe and America.

that histories of separate subjects will gradually be written similar to those of the late Dr Todhunter on the theory of attractions and the calculus of probabilities; whenever that is done, it will be possible from those separate works to construct a general history of the mathematics of this century. Something of this kind will be found in the annual volumes of the British Association, which contain a number of reports on the progress in several of the branches of modern mathematics.

I have tried to arrange the mathematicians mentioned above according to the subjects in connection with which they are best known, arranging the latter in the following order: Elliptic and Abelian functions, Theory of numbers, Higher algebra, Modern geometry, Analytical geometry, Analysis, Astronomy, and Physics.

I should add that where a writer is the author of numerous memoirs and not of any single treatise on a subject I have generally merely noted the fact that he has written on it; and I would refer any one who wishes for more details to the invaluable classified catalogue of all the scientific papers contributed during this century to any society or journal which has been compiled by the Royal Society of London.

Elliptic and Abelian functions.

In discussing Legendre's work on elliptic functions I mentioned (see p. 395) that he lived to see his methods of treatment superseded by those of Abel and Jacobi. The researches of the two last-named mathematicians formed the starting-point for a number of writers of whom the most prominent are perhaps Riemann, Weierstrass, Rosenhain, Henry Smith, Göpel, Cayley, Hermite, Königsberger, and Halphen. I add here a few notes on Abel, Jacobi, and Riemann; and in the next section give a few lines on the work of Henry Smith.

Niels Henrik Abel was born at Findoë in Norway in 1802 and died at Arendal in 1829, at the age of twenty-six. His memoirs on elliptic functions which were originally published in Crelle's Journal treat the subject from the point of view

of the theory of equations and algebraic forms, a treatment to which his researches naturally led him. The important and very general result known as Abel's theorem, which was subsequently extended by Riemann, was sent to the French Academy in 1828, but was not read or published till twenty years later. The name of Abelian function has been given to the higher transcendents of multiple periodicity which were first discussed by him. As illustrating his wonderful fertility of ideas I may in passing notice his celebrated demonstration that it is impossible to solve a quintic equation by means of radicals; this theorem was the more important since it definitely limited a field of mathematics which had previously attracted numerous writers. Two editions of Abel's works have been published, of which the last, edited by Sylow and Lie and issued at Christiania in 2 volumes in 1881, is far the best. His life has been recently written by Bjerknes and published at Stockholm in 1880.

Karl Gustav Jacob Jacobi, born of Jewish parents at Potsdam on Dec. 10, 1804 and died at Berlin on Feb. 18, 1851, was educated at the university of Berlin where he obtained the degree of doctor of philosophy in 1825. In 1827 he became extraordinary professor of mathematics at Königsberg, and in 1829 was promoted to be an ordinary professor; this chair he occupied till 1842, when the Prussian government gave him a pension, and he moved to Berlin where he continued to live till his death in 1851. His most celebrated investigations are those on elliptic functions, the modern notation in which is due to him, and the theory of which he established simultaneously with Abel but independently of him. These are given in his treatise Fundamenta nova theoriæ functionum ellipticarum, Königsberg, 1829, and in some later papers in Crelle's Journal. The correspondence between Legendre and Jacobi on elliptic functions edited by Borchardt is given in Crelle's Journal for 1875, and has been reprinted in vol. I. of Jacobi's collected works. Jacobi, like Abel, saw that the importance of the subject was not that it was a group of theorems on integration, but that it introduced a new kind of

function, namely one of double periodicity; hence he paid particular attention to the theory of the theta-function. The following passage (on p. 87 of vol. I. of his collected works) in which he explains this view is sufficiently interesting to deserve textual reproduction, "E quo, cum universam, quae fingi potest, amplectatur periodicitatem analyticam elucet, functiones ellipticas non aliis adnumerari debere transcendentibus, quae quibusdam gaudent elegantiis, fortasse pluribus illas aut maioribus, sed speciem quandam iis inesse *perfecti et absoluti.*"

Among Jacobi's other investigations I may specially single out his papers on determinants, which did a great deal to bring them into general use; and particularly his invention of the Jacobian, that is of the functional determinant formed by the n^2 partial differential coefficients of the first order of n given functions of n independent variables. I ought also to mention his papers on Abelian transcendents; his investigations on the theory of numbers, these latter being founded on those of Gauss; and his numerous memoirs on the planetary theory and other particular dynamical problems, in the course of which he added considerably to the theory of differential equations. Most of these researches are included in his Vorlesungen über Dynamik, edited by Clebsch, Berlin, 1866. His collected works were published at Berlin, 2nd edition, 1881.

Georg Friederich Bernhard Riemann, born at Breselenz on Sept. 17, 1826 and died at Selasca on July 20, 1866, studied at Göttingen under Gauss, and subsequently at Berlin under Jacobi, Dirichlet, Steiner, and Eisenstein, all of whom were professors there at the same time. His earliest paper was in 1850 on functions of a complex variable. This was succeeded in 1854 by one on the hypotheses on which geometry is founded. His chief memoirs are on elliptic functions, the theory of numbers, and the fundamental conceptions of geometry, but he also wrote on physical subjects. It is hardly too much to say that in his memoir on elliptic functions in Borchardt's Journal for 1857 he did for the Abelian functions what Abel had done for

the elliptic functions, and it is this perhaps that will constitute one of his chief claims to future distinction. His short tract of eight pages on the number of primes which lie between two given numbers is one of the most striking instances of his genius and analytical powers. Legendre had previously shewn (Th. des Nom. § 404) that the number of primes less than n is very approximately $n/(\log n - 1\cdot08366)$; but Riemann went further, and this tract contains all that has yet been done in connection with a problem of so obvious a character that it suggested itself to every mathematician who considered the theory of numbers and yet which overtaxed the powers of all his predecessors, including even Lagrange and Gauss. His paper on the fundamental conceptions of geometry has excited much interest and discussion. His collected works, edited by Weber and prefaced by an account of his life by Dedekind, were published at Leipzig in 1876.

Among other and more recent works I may specially mention the following. The Zur Theorie der Abel'schen Integrale by **Weierstrass**, 1849, which with other papers and lectures by the same author has created a new development of the subject. A memoir by **Rosenhain** in the transactions of the Berlin Academy for 1848 on the double theta function; his De integralibus functionum algebraicarum, 1844; his Ueber die hyperelliptischen Transcendenten, 1844; and his Sur les fonctions de deux variables et à quatre périodes, 1850. The researches of **Henry Smith** which are chiefly on the theta and omega functions, will be found in his collected works now being issued by the university press at Oxford. The most important of those of **Göpel** are on the hyperelliptic (double theta) functions: for further details see a note by Jacobi in vol. 35 of Crelle's Journal. The most important of those of **Cayley** are on the connection between Legendre and Jacobi, and will be found in his collected works which the university of Cambridge are now preparing. The researches of **Hermite** are mostly concerned with the transformation theory, a subject which he almost created, and with the

connection between the methods and results of Weierstrass and Jacobi. The transformation of the double theta function has also been considered by **Königsberger**. The investigations of **Halphen** (which are largely founded on Weierstrass' work) are included in his Fonctions elliptiques in 3 vols. of which only the first has been yet (1888) issued. I should add that the text-book on Elliptic functions by **Briot** and **Bouquet**, 2nd ed. 1875, contains a clear account of the subject as it exists at present, developed from the point of view of the complex variable.

The theory of numbers.

I have already mentioned Gauss as having been engaged as early as 1801 in researches on the theory of numbers—researches which were carried on at the same time as but independently of those of Legendre. Gauss' work served as the starting-point for a school of writers of whom some of the most celebrated members are Jacobi, Dirichlet, Cauchy, Liouville, Eisenstein, Henry Smith; and among living mathematicians Kummer, Kronecker, Hermite, Dedekind, and Tchebycheff. Interest in investigations similar to those of these writers seems to have recently flagged, and it is possible that the subject may be better approached on other lines. I add here a few notes on the work of Gauss, Dirichlet, and Henry Smith: the writings of Jacobi are briefly alluded to on p. 417, and those of Cauchy on p. 436.

Karl Friederich **Gauss** was born at Brunswick, April 23, 1777 and died at Göttingen on Feb. 23, 1855. His father was a bricklayer, and Gauss was indebted for a liberal education (much against the will of his parents) to the notice which his talents procured from the reigning duke. In 1799 Gauss published his proof that every algebraical equation has a root of the form $a+bi$. In 1801 this was followed by his Disquisitiones arithmeticæ on the theory of numbers, and which forms the first volume of his collected works. This was sent to the French Academy and rejected with a sneer which, even if the book had been as worthless as the referees be-

lieved, would have been unjustifiable: Gauss was deeply hurt, and his reluctance to publish his investigations is chiefly attributable to this unfortunate incident. The next year he calculated the elements of the planet Ceres from data which had been previously supposed to be insufficient. The attention excited by these investigations procured for him in 1807 the appointment of director of the Göttingen observatory, an office which he retained to his death; and it is said that after his appointment he never slept away from his observatory except on one occasion when he attended a scientific congress at Berlin. In 1809 he published at Hamburg his Theoria motus corporum cœlestium, a work which largely contributed to the improvement of practical astronomy: and on the same subject but connected with observations in general we have his memoir Theoria combinationis observationum erroribus minimis obnoxia, with a second part and a supplement. His first paper on the theory of magnetism entitled Intensitas vis magneticæ terrestris ad mensuram absolutam revocata, was published in 1833. A few months afterwards he together with Weber invented the declination instrument and the bifilar magnetometer: and in the same year they erected at Göttingen a magnetic observatory free from iron (as Humboldt and Arago had previously done on a smaller scale) where they made magnetic observations, and in particular shewed that it was possible and practicable to send telegraphic signals. In connection with this observatory Gauss founded the association of Magnetische Verein with the object of securing continuous observations at fixed times. The volumes of their publications, Resultate aus der Beobachtungen des Magnetischen Vereins, for 1838 and 1839, contain the two important memoirs by Gauss entitled Allgemeine Theorie der Erdmagnetismus and Allgemeine Lehrsätze, on the theory of forces attracting according to the inverse square of the distance. He co-operated in the Danish and Hanoverian geodetical operations which lasted from 1821 to 1848; and in connection with these he wrote in 1843 and 1846 the two

possible to find a square number which when divided by p leaves a remainder n, in other words n is a quadratic residue of p: when the remainder is -1 there exists no such square number, and n is a non-residue of p. The law of quadratic reciprocity is expressed by the theorem that if a and b are any odd primes then

$$\left(\frac{a}{b}\right)\left(\frac{b}{a}\right)=(-1)^{\frac{1}{4}(a-1)(b-1)};$$

thus if b is a residue of a then a is also a residue of b, unless both of the primes a and b are of the form $4m+3$.

In other words, if a and b be odd primes we know that

$$a^{\frac{1}{2}(b-1)}\equiv\pm1 \pmod b, \text{ and } b^{\frac{1}{2}(a-1)}\equiv\pm1 \pmod a;$$

but by Legendre's law the two ambiguities will either be both positive or both negative, unless a and b are both of the form $4m+3$. Thus if one odd prime is a non-residue of another then the latter will be a non-residue of the former. Gauss and Kummer have subsequently proved similar laws of cubic and biquadratic reciprocity; and an important branch of the theory of numbers has been based on these researches. This work also contains the useful theorem by which, when it is possible, an indeterminate equation of the second degree can be reduced to the form $ax^2+by^2+cz^2=0$, and a discussion of numbers which can be expressed as the sum of three squares.

The *Exercises de calcul intégral* was published in three volumes 1811, 1817, 1826. Of these the third and most of the first are devoted to elliptic functions: the bulk of this being ultimately included in the *Fonctions elliptiques*. The contents of the remainder of the treatise are of a very miscellaneous character: they include integration by series, definite integrals, and in particular an elaborate discussion of the Beta and the Gamma functions.

The *Traité des fonctions elliptiques* was issued in two volumes in 1825 and 1826 and is the most important of Legendre's works.

papers Ueber Gegenstände der höhern Geodäsie. In 1840 he wrote the Dioptrische Untersuchungen. Of the remaining memoirs in pure mathematics the most remarkable are those on the theory of biquadratic residues (in which the notion of complex numbers of the form $a+bi$ was first introduced into the theory of numbers) in which are included several tables, and among others one of the number of the classes of binary quadratic forms: that relating to the proof of the theorem that every numerical equation has a real or imaginary root: that entitled Summatio quarundam serierum singularium: and lastly one on hypergeometric series, and another on interpolation. We have also the memoir Allgemeine Auflösung, on the graphical representation of one surface upon another; and the Disquisitiones generales circa superficies curvas et series infinitas: the latter contains a discussion of the gamma-function. In the theory of attractions we have a paper on the Attraction of homogeneous ellipsoids: the already-mentioned tract Allgemeine Lehrsätze, on the theory of forces attracting according to the inverse square of the distance; and the memoir Determinatio attractionis, in which a planetary mass is considered as distributed over its orbit according to the time in which each portion of the orbit is described, and the question is to find the attraction of such a ring.

Leaving out of account his theory of magnetism and his papers on the practical sides of astronomy and magnetism, his most celebrated work is his Disquisitiones arithmeticæ. This and Legendre's Théorie des nombres remain standard works on the theory of numbers. But just as in his discussion of elliptic functions Legendre showed himself unable to rise to the conception of a new subject, and confined himself to regarding their theory as a chapter in the integral calculus, so he treated the theory of numbers as a chapter in algebra. Gauss however realized that the theory of discrete magnitudes or higher arithmetic was of a totally different kind to that of continuous magnitudes or algebra, and he invented the notation and

Gauss' work on the theory of numbers was supplemented by that of **Jacobi**, who first proved the law of cubic reciprocity; discussed the theory of residues; and in his *Canon arithmeticus* gave a table of residues of prime roots. The subject was next taken up by Lejeune Dirichlet, who is known rather as the expounder of Gauss than for his own original investigations, valuable though some of these are. *Peter Gustav Lejeune* **Dirichlet** was born at Düren on Feb. 13, 1805 and died at Göttingen on May 5, 1859. He held successively professorships at Breslau and Berlin, and on Gauss' death in 1855 was appointed to succeed him at Göttingen. He intended to finish Gauss' incomplete works, for which he was admirably fitted, but his early death prevented his effecting this. Dirichlet wrote numerous memoirs, many of which served to translate Gauss' works into a better and more intelligible notation. Of his original work the most celebrated is that on the determination of means with applications to the distribution of prime numbers. His papers on the theory of numbers have been edited by R. Dedekind, 3rd edition, Brunswick, 1879–81. His work on the theory of the potential has been edited by F. Grube, 2nd edition, Leipzig, 1887. There is a short note on some of his investigations by C. W. Borchardt in vol. 57 of Crelle's *Journal*.

Of all the school founded by Gauss no one however can compare for originality and power with Henry Smith. *Henry John Stephen* **Smith** was born in London on Nov. 2, 1826 and died at Oxford on Feb. 9, 1883. He was educated at Rugby, and at Balliol College, Oxford, of which latter society he was a fellow; and in 1861 he was elected Savilian professor of geometry at Oxford, where he resided till his death.

The subject in connection with which Smith's name will always be specially remembered is the theory of numbers, and to this he devoted the years from 1854 to 1864. The results of his historical researches were given in his report published in parts in the British Association volumes from 1859 to 1865, which contains an account of everything that had been done on the subject

to that time together with some additional matter. The chief outcome of his own original work on the subject is included in two memoirs printed in the Phil. Trans. for 1861 and 1867; the first being on linear indeterminate equations and consequences, and the second on the orders and genera of ternary quadratic forms. Pure mathematics is divisible into two great branches, the theory of numbers, or "arithmetic," i.e. the theory of discrete magnitude, and algebra, i.e. the theory of continuous magnitude, the aims and methods of the two subjects being quite distinct. A characteristic of Smith's work, no less than of Gauss', is the "arithmetical" mode of treatment that runs through the whole of it, no matter what the subject; and his great command over the processes of that science is everywhere conspicuous. The "algebraical" method of treatment is similarly illustrated by the works of Euler in the last century, or of Cayley in more recent times.

The two great divisions into which the theory of numbers may be divided are the theory of congruences and the theory of forms. The solution of the problem of the representation of numbers by binary quadratic forms is one of the great achievements of Gauss, and the fundamental principles upon which the treatment of such questions must rest were given by him in the Disquisitiones arithmeticæ. Gauss there added some results relating to ternary quadratic forms, but the extension from two to three indeterminates was the work of Eisenstein, who in his memoir Neue Theoreme der höheren Arithmetik, defined the ordinal and generic characters of ternary quadratic forms of an uneven determinant; and, in the case of definite forms, assigned the weight of any order or genus; but he did not consider forms of an even determinant, nor give any demonstrations of his work. These omissions were supplied by Smith in his great memoir on the subject, which contains a complete classification of ternary quadratic forms. Smith, however, did not confine himself to the case of three indeterminates, but succeeded in establishing the principles on which the extension to the general case of n indeterminates

depends, and obtained the general formulæ; thus effecting the greatest advance made in the subject since the publication of Gauss' work.

A brief account of Smith's methods and results appeared in the Proceedings of the Royal Society (vol. XIII., 1864, pp. 199—203, and vol. XVI., 1868, pp. 197—208). In the second of these notices, he remarks that the theorems relating to the representation of numbers by four squares and other simple quadratic forms, are deducible by a uniform method from the principles indicated in the paper, as also are the theorems relating to the representation of numbers by six and eight squares. He then proceeds, "As the series of theorems relating to the representation of numbers by sums of squares ceases, for the reason assigned by Eisenstein, when the number of squares surpasses eight, it is of some importance to complete it. The only cases which have not been fully considered are those of five and seven squares. The principal theorems relating to the case of five squares have indeed been given by Eisenstein (Crelle's Journal, vol. XXXV. p. 368); but he has considered only those numbers which are not divisible by any square. We shall here complete his enunciation of those theorems, and shall add the corresponding theorems for the case of seven squares."

This paper was the occasion of a dramatic incident in the history of mathematics. The class of theorems in question (viz. the number of representations of a number as a sum of squares) had been shown by Eisenstein to be limited to eight squares. The solutions in the cases of two, four, and six squares may be obtained by means of elliptic functions, i.e. by purely algebraic methods, but the cases in which the number of squares is uneven involve the special processes peculiar to the theory of numbers. Eisenstein had given the solution in the case of three squares, and he also left a statement of the solution he had obtained in the case of five squares. His results, however, were published without demonstration, and only apply to numbers having a particular form. The com-

plete solution was indicated by Smith in the above-mentioned papers. Fourteen years later, in ignorance of Smith's work, the demonstration and completion of Eisenstein's theorems for five squares were set by the French Academy as the subject of their "Grand Prix des Sciences Mathématiques." Smith wrote out the demonstration of his general theorems so far as was required to prove the results in the special case of five squares, and only a month after his death, in March 1883, the prize was awarded to him, another prize being also awarded to M. Minkowski. No episode could bring out in a more striking light the extent of Smith's researches than that a question of which he had given the solution in 1867 as a corollary from general formulæ which governed the whole class of investigations to which it belonged should have been regarded by the French Academy as one whose solution was of such difficulty and importance as to be worthy of their great prize. It is perhaps even more astonishing that they should have known so little of contemporary English and German researches on the subject as to be unaware that the result of the problem they were proposing was at the time lying in their own library.

Smith was also the author of important papers in which he succeeded in extending to complex quadratic forms many of Gauss' investigations relating to real quadratic forms. In 1868 he was awarded the Steiner prize of the Berlin Academy for a geometrical memoir "Sur quelques problèmes cubiques et biquadratiques." In a paper which he contributed to the *Atti* of the Accademia dei Lincei for 1877 he established a very remarkable analytical relation connecting the modular equation of order n and the theory of binary quadratic forms belonging to the positive determinant n. In this paper the modular curve is represented analytically by a curve in such a manner as to present an actual geometrical image of the complete systems of the reduced quadratic forms belonging to the determinant, and a geometrical interpretation is given to the ideas of *class, equivalence,* and *reduced form.*

He was led by his researches on the theory of numbers

to the theory of elliptic functions; and on this subject the results he arrived at, especially on the theory of the theta and omega functions, are of great importance.

Smith's collected mathematical works, edited by Dr Glaisher of Trinity College, Cambridge, will shortly be issued by the Oxford University Press.

The researches of **Cauchy** on the theory of numbers are included in the complete edition of his works now being issued by the French Government. Many of them deal with the expression of quadratic binomials in particular forms, but their miscellaneous character renders it difficult to describe them briefly. Most of the investigations of **Liouville** are on the expression of numbers in special forms. Those of **Eisenstein** are in his memoir Untersuchungen über die Cubischen Formen mit zwei Variabeln, in vol. 27 of Crelle's Journal; in his Mathematische Abhandlungen, 1847; and his Die Vergleichung etlicher ternären quadratischen Formen (with tables), 2 vols. 1851. Those of **Kummer** are in his De residuis cubicis, 1842; De numeris complexis..., 1844; his Ueber die Reciprocitäts-gesetze..., 1859, to which an appendix with additions was added in 1862; and also some memoirs on hypergeometric series. Those of **Kronecker** in his De unitatis complexis, Berlin, 1845, and numerous papers in the Berlin Academy on ternary and quadratic forms. Those of **Hermite** are mostly on ternary forms. The most important researches of **Dedekind** are given in an appendix to his edition of Lejeune Dirichlet's writings and are on ideal primes. Those of **Tchebycheff** are on the number of primes between given limits.

Higher algebra.

The theory of numbers may be called the theory of higher arithmetic; and while one group of writers devoted themselves to that, another group has immensely extended the range of modern algebra. Chief among these are Hamilton, Cauchy, Galois, Boole, Borchardt, Eisenstein, and de Morgan; also

among more recent writers Cayley, Sylvester, Salmon, Serret, Jordan, Hermite, Betti, Brioschi, Aronhold, Poincaré, Gordan, Clebsch, and Macmahon. I add a few notes on the writings of Hamilton, Boole, and de Morgan.

Sir William Rowan **Hamilton** was born of Scotch parents in Dublin on Aug. 4, 1805 and died there on Sept. 2, 1865. His education, which was carried on at home, seems to have been singularly discursive; under the influence of an uncle who was a good linguist he first devoted himself to linguistic studies; by the time he was seven he could read Latin, Greek, French, and German with facility; and when thirteen he was able to boast that he was familiar with as many languages as he had lived years. It was about this time that he came across a copy of Newton's Universal Arithmetic. This was his introduction to modern analysis; and he soon mastered the elements of analytical geometry and the calculus. He then read the Principia; and next the four volumes then published of Laplace's Mécanique céleste. In the latter he detected a mistake, and his paper on the subject written in 1823 placed him at once in the front rank of mathematicians. In the following year he entered at Trinity College, Dublin. His university career is unique, for the chair of astronomy becoming vacant in 1827, while he was yet an undergraduate, he was asked by the electors to stand for it, and was elected unanimously, it being understood that he should be left free to pursue his own line of study.

His earliest paper written in 1824 was on optics, and was published in 1828 under the title of A theory of systems of rays, to which two supplements, written in 1831 and 1832, were afterwards added; in the latter of these the phenomenon of conical refraction is predicted. This was followed by a paper in 1827 on the principle of Varying Action, and in 1834 and 1835 by memoirs on a General method in dynamics. His lectures on Quaternions were published in 1852. Amongst his numerous papers, those on the form of the solution of the general algebraic equation of the fifth degree (which cannot be expressed in terms of the more elementary operations and

functions); on fluctuating functions; on the hodograph; and on the numerical solution of differential equations, have left the deepest mark on the subjects they respectively deal with. Lastly his Elements of quaternions were issued in 1866. Of this a competent authority says that the methods of analysis here given are as great an advance over those of analytical geometry, as the latter were over those of Euclidean geometry.

Hamilton was painfully fastidious on what he published, and he has left an immense collection of manuscript books which are in the library of Trinity College, Dublin, and may it is hoped be some day printed. For further details his life by R. P. Graves, Dublin, 1882, may be consulted.

George Boole, born at Lincoln on Nov. 2, 1815 and died at Cork on Dec. 8, 1864, was a self-educated and most original mathematician. His chief works are one on differential equations in 2 vols. 1859—65, and another on finite differences published in 1860. The theory of covariants has grown out of his papers on linear transformations; and he developed a system of non-commutative algebra.

Augustus de Morgan, born in Madura (Madras) in June, 1806 and died in London on March 18, 1871, was educated at Trinity College, Cambridge, but in the then state of the law was (as a unitarian) unable to stand for a fellowship. In 1828 he became professor at the newly-established university of London, which is the same institution as that which now forms University College. Here (except for five years from 1831 to 1835) he taught continuously till 1867, and through his works and pupils exercised a wide influence on English mathematicians of the present day. The London Mathematical Society was largely his creation, and he took a prominent part in the proceedings of the Royal Astronomical Society.

He was perhaps more deeply read in the philosophy and history of mathematics than any of his contemporaries, but the results are given in scattered and almost inaccessible articles which well deserve collection and re-publication. A list of them is given in his life. The best known of his works are

the memoirs on the foundation of algebra (Camb. Phil. Trans. vols. VIII. and IX.); his great treatise on the differential calculus published in 1842, which is a work of the highest ability; and his articles on the calculus of functions and on the theory of probabilities (Encycl. Metropol.). The calculus of functions contains an investigation of the principles of symbolic reasoning, but the applications deal with the solution of functional equations rather than with the general theory of functions. The article on probabilities gives a very clear analysis of the mathematics of the subject to the time at which it was written. For further details of his life his memoir by his widow S. E. de Morgan, London, 1882, may be consulted.

Cauchy's work on algebra is alluded to a few pages later; it will be enough here to say that it includes numerous valuable papers on the theory of equations, and the theory of functions. Most of **Galois'** papers deal with the former of these subjects. Among other writers on the subject of higher algebra I should mention the following. **Borchardt**, who in particular discussed generating functions in the theory of equations, and arithmetic-geometric means; a collected edition of his works, edited by G. Hettner, has recently been issued by the Prussian government. **Eisenstein**, who gave a criterion for distinguishing whether a given series represents an algebraical or a transcendental function. **Cayley**, whose classical memoirs on quantics (binary and ternary forms) and researches on non-commutative algebras, especially on matrices, will be found in the collected edition of his works now being issued by the University Press at Cambridge. **Sylvester**, from among whose numerous memoirs I may in particular single out those on canonical forms, the theory of contravariants, on reciprocants, the theory of equations, and both that on Newton's rule. Sylvester is also the creator of the language and notation of considerable parts of this as well as of other subjects on which he has written. **Jordan**, who has written on the theory of substitutions with special applications to differential equations. **Hermite**, who has

in particular discussed the theory of associated covariants in binary quantics, the theory of ternary quantics, and the quintic; and who has applied elliptic functions to the solution of the quintic equation. **Betti** and **Brioschi**, who also have discussed binary quantics in detail. **Aronhold**, who developed symbolic methods especially in connection with ternary quantics; this was done concurrently but independently of Cayley's work on the same subject. **Poincaré**, whose most characteristic investigations are connected with the theory of functions, with special applications to differential equations. **Gordan**, who has discussed the theory of forms, and shown that there are only a finite number of concomitants of quantics: an edition of his work on invariants (determinants and binary forms) edited by Kerschensteiner was issued at Leipzig in 2 vols. in 1887. **Clebsch**, who also has independently investigated the theory of forms in some papers which will be found in his collected works recently edited by Lindemann. And lastly **Macmahon**, who has written on the connection of symmetric functions, the derivation of invariants and covariants from elementary algebra, and the concomitants of binary forms. No account of contemporary writings on this subject would be complete without a reference to the admirable text-books produced by **Salmon** in his Higher algebra, and by **Serret** in his Cours d'Algèbre supérieure, in which the chief discoveries of their respective authors are embodied. An admirable historical summary of the theory of the complex variable is given in **Hankel's** Vorlesungen über die complexen Zahlen, Leipzig, 1867.

Modern geometry.

Modern geometry has been considerably developed in this period. The chief works, after those of Monge in 1800, Carnot in 1803, and Poncelet in 1822, are the Barycentrischer Calcul published in 1826 by *August Ferdinand* **Möbius**, who was one of the best known of Gauss' pupils, and who also continued Gauss' researches on astronomy and mechanics: **Steiner's** Abhängigkeit geometrischer Gestalten, 1832, which

contains the first full discussion of the projective relations between rows, pencils, &c.: **Von Staudt's** Geometrie der Lage, 1847, and Beiträge zur Geometrie der Lage, 1856—60, in which a system of geometry is built up from the beginning without any reference to number so that ultimately a number itself gets a geometrical definition: **Cremona's** Introduzione ad una teoria geometrica delle curve piane, 1862, and its continuation Preliminari di una teoria geometrica delle superficie: and lastly **Sir Robert Ball's** Theory of Screws, London, 1876. The first and last of these works though nominally on mechanics contain many theorems of great geometrical interest. As more elementary books, I may mention **Chasles'** Traité de géométrie supérieure, 1852; **Steiner's** Vorlesungen über synthetische Geometrie, edited by Geiser and Schröder, 1867; **Cremona's** Éléments de géométrie projective, translated into French by E. Dewulf, 1875; and **Reye's** Die Geometrie der Lage, 1882. In addition to these the true foundation of geometry has been considered by **Grassmann** in his Ausdehnungslehre, 1844; and in later times by **Riemann** (see p. 419), and by **Helmholtz**. I add a few notes on the works of Chasles and Clifford.

Michel **Chasles**, who was born at Epernon on Nov. 15, 1793 and died at Paris on Dec. 18, 1880, devoted himself to the study of modern geometry. His chief works were the Aperçu historique, 1837, which is continued in the Rapport sur les progrès de la géométrie, 1870; his Higher geometry, 1852; and his Conic sections, 1865. The first of these constitutes his best-known claim to distinction. It is an interesting summary of the history of geometry, though the subject in modern times is treated from an exclusively French point of view, and nearly all the German works on it are neglected or depreciated. The history of the Pascal forgeries illustrates the same desire to trace all discoveries to French sources.

The early death of Clifford prevented his taking that position which the originality of his works seemed to promise. The varied character of his writings makes it difficult to

this century to devote himself to the development of analytical geometry. His chief results are embodied in his work entitled A treatise on some new geometrical methods. The researches of **MacCullagh**, which include some valuable discoveries on the theory of quadrics, will be found in his collected works edited by Jellett and Haughton, Dublin, 1880. *Julius* **Plücker** was born at Elberfeld on July 16, 1801 and died at Bonn on May 22, 1868. After lecturing at Bonn, Berlin, and Halle he obtained in 1836 the chair of mathematics at Bonn, and devoted himself chiefly to the study of a geometry in which the line is the element in space, and the theory of congruences and complexes. His equations connecting the singularities of curves are well known. In 1847 he exchanged his chair for one of physics, and his subsequent researches were on spectra and magnetism. He contributed numerous memoirs to scientific journals. His chief works are Analyseos applicatio ad geometriam et mechanicam, 1821; Analytisch-geometrische Entwickelungen, 1831; System der analytischen Geometrie, 1835; Theorie der algebraischen Curven, 1839; System der Geometrie des Raumes in neuer analytischer Behandlungsweise 1846; Neue Geometrie des Raumes, gegründet auf die Betrachtung der geraden Linie als Raumelement, 1868. The majority of the memoirs by **Cayley** are on the theory of curves and surfaces and will be found in the edition of his collected works now being issued by the University Press at Cambridge; the most remarkable of those of **Hesse** are on the plane geometry of curves; of those of **Darboux** on the geometry of surfaces; of those of **Halphen** on the singularities of surfaces; and of those of **Zeuthen** and **Schubert** also on the singularities of curves and surfaces. **Clebsch** has applied Abel's theorem to geometry, and **Klein** has created the theory of polyhedral functions. Among more recent works or text-books are **Clebsch's** Vorlesungen über Geometrie 1875; and **Salmon's** Conic Sections, Geometry of three dimensions, and Higher plane curves; in which the chief discoveries of those writers are embodied.

Analysis.

Among those who have extended the range of analysis, or whom it is difficult to place in any of the preceding categories are the following, whom I place in alphabetical order. **Appell,** who has written on the theory of functions and differential equations. **Beltrami,** who has made a special study of the theory of hyperbolic space. **Bertrand,** whose work on theoretical dynamics no less than his text-book on the calculus is of a high order of excellence. **Cauchy,** who wrote on most of the subjects of pure mathematics (and especially of analysis) which were debated in the first half of this century. **Crofton,** who has written on local probability. **Darboux,** who has written largely on the subject of differential equations. **Dedekind,** who is the author of a remarkable memoir on the vibrations of a liquid ellipsoid, which is treated as a problem in pure mathematics. **Frobenius,** most of whose papers are either on differential equations or elliptic analysis. **Fuchs,** who has greatly developed the theory of differential equations and especially of linear equations. **Halphen,** who has also made a special study of differential equations and also of differential invariants (reciprocants). **Jacobi,** to whose work on determinants and differential equations allusion has already been made. **Jordan,** who has applied the theory of algebraic substitutions to differential equations. **Königsberger,** most of whose papers deal with different points in the theory of differential equations. **Lie,** who has investigated the theory of partial differential equations of the first order. **Mittag-Leffler,** who has greatly developed the theory of functions. **Poincaré,** most of whose memoirs deal either with the subject of differential equations or with the theory of functions. **Sylvester,** who among other subjects has written on reciprocants. **Weierstrass,** who has created the modern theory of functions, and in particular has discussed in great detail the theory of analytical functions, elliptic functions, hyperelliptic functions, and I believe also the calculus of variations.

I add a few notes on the work of Cauchy. *Augustin Louis* **Cauchy,** who was born at Paris on Aug. 21, 1789 and died at

Sceaux on May 25, 1857, was educated at the Polytechnic school, which was the nursery of so many French mathematicians of that time, and adopted the career of the ponts et chaussées. His earliest mathematical paper was one on polyhedra in 1811. Legendre thought so highly of it, that he asked Cauchy to attempt the solution of an analogous problem which had baffled previous investigators; and his advice was justified by the success of Cauchy in 1812. Memoirs on analysis and the theory of numbers presented in 1813, 1814, and 1815 showed that his ability was not confined to geometry alone: in one of these papers he generalized some results which had been established by Gauss and Legendre; in another of them he gave a theorem on the number of values which an algebraical function can assume when the literal constants it contains are interchanged. It was the latter theorem that enabled Abel to shew that an algebraic equation of a degree higher than the fourth cannot in general be solved by the use of algebraical expressions.

On the restoration in 1816 the French Academy was purged, and in spite of the indignation and scorn of French scientific society, Cauchy accepted a seat which was procured for him by the expulsion of Monge. He was also at the same time made professor at the Polytechnic; and his lectures there form the subject of his text-books on Algebraic analysis, the Differential calculus, and the Theory of curves. On the revolution in 1830 he went into exile, and was first appointed professor at Turin, whence he soon moved to Prague to undertake the education of the Comte de Chambord. He returned to France in 1837; and in 1848 and again in 1851 by special dispensation of the emperor was allowed to occupy a chair of mathematics without taking the oath of allegiance.

His activity was prodigious, and from 1830 to 1859 he published in the transactions of the Academy or the Comptes Rendus over 600 original memoirs and about 150 reports. In most of them the feverish haste with which they were thrown off is too visible; and many are marred by obscurity, repetition of old results, and blunders.

Among the more important of his researches are the determination of the number of real and imaginary roots of any algebraic equation; his method of calculating these roots approximately; his theory of the symmetric functions of the coefficients of equations of any degree; his *à priori* valuation of a quantity less than the least difference between the roots of an equation; and his papers on determinants in 1841 which brought them into general use. Cauchy also did a great deal to reduce the art of determining definite integrals to a science, but this branch of the integral calculus still remains without much system or method. The method for finding the principal values of integrals and the calculus of residues were invented by him. His proof of Taylor's theorem seems to have originated from a discussion of the double periodicity of elliptic functions. The method of showing a connection between different branches of a subject by giving imaginary values to independent variables is largely due to him. He also gave a direct analytical method for determining planetary inequalities of long period; and to physics he contributed an *à priori* method for finding the quantity of light reflected from the surfaces of metals as well as other papers on optics.

For further details see La vie et les travaux de Cauchy by Valson, Paris, 1868. A complete edition of his works is now being issued by the French government.

Astronomy.

Among those who in this century have devoted themselves to the study of the comparatively limited subject of theoretical astronomy the name of Gauss is one of the most prominent; to his work I have already alluded. The best known of his contemporaries was *Friederich Wilhelm* Bessel, who was born at Minden on July 22, 1784 and died at Königsberg on March 17, 1846. Bessel commenced his life as a clerk on board ship, but in 1806 he became an assistant in the observatory at Lilienthal, and was thence in 1810 promoted to be director of the new

Prussian observatory at Königsberg where he continued to live during the remainder of his life. He introduced Bessel's functions into pure mathematics. His collected works and correspondence have been edited by Engelmann and published in 4 volumes at Leipzig, 1875—82. Other well-known astronomers of a slightly later date are **Plana**, whose work on the motion of the moon was published in 1832; **Count Pontécoulant**; **Delaunay**, whose work on the lunar theory and (incomplete) lunar tables are among the great astronomical achievements of this century; the former indicates the best method yet suggested for the analytical investigation of the whole problem: and **Hansen**, who compiled the lunar tables published in London in 1857, and elaborated the most delicate methods yet known for the determination of lunar and planetary perturbations; for an account of his numerous memoirs the obituary notice in the transactions of the Royal Society of London for 1876—77 may be consulted.

Among the astronomical events of this century the discovery of the planet Neptune by Leverrier and Adams is one of the most striking. *Urbain Jean Joseph* **Leverrier**, born at St Lô on March 11, 1811 and died at Paris on Sept. 23, 1877, is amongst the greatest of modern astronomers. His earliest researches in astronomy were communicated to the Academy in 1839: in these he calculated within much narrower limits than Laplace had done the extent within which the inclinations and eccentricities of the planetary orbits vary. The independent discovery in 1846 by Leverrier and Adams of the planet Neptune by means of the disturbance it produced on the orbit of Uranus will always be one of the most striking events in physical astronomy. In 1855 Leverrier succeeded Arago as director of the Paris observatory, and reorganized it in accordance with the requirements of modern astronomy. He now set himself the task of discussing all the theoretical investigations of the planetary motions and of revising all tables which involved them. He just lived long enough to

sign the last proof-sheet of this invaluable work. For further details of his life see Bertrand's éloge in vol. XLI. of the Mém. de l'Acad.; and for an account of his work see Adams' address in vol. XXXVI. of the Monthly notices of the Royal Astronomical Society.

Among living astronomers the names of J. C. **Adams**, the Lowndean professor at Cambridge and co-discoverer of Neptune with Leverrier, who wrote the celebrated paper on the secular acceleration of the moon's mean motion (Phil. Trans. 1853); G. W. **Hill**, the author of a recent work on the motion of the moon and of an investigation on the motion of a planet's perigee under certain conditions; and G. H. **Darwin**, the Plumian professor at Cambridge and author of several papers on the tides upon viscous spheroids which were published in the Philosophical Transactions, will occur to all interested in the subject.

Mathematical physics.

Mathematical physics lies outside the limits I have laid down for myself in this book, but no account of this century would be other than misleading which failed to call attention to the energy and skill shown in applying mathematics to numerous problems in nature which were but recently outside the range of exact reasoning; and it is certain that whenever the history of this time comes to be written the names of men like Faraday, Clerk Maxwell, Helmholtz, and above all Sir William Thomson will occupy a prominent position.

Amongst the chief writers on mathematical physics I must specially mention the following whose names are here arranged alphabetically. **Boltzmann**, who has greatly extended the kinetic theory of gases, and done something to bring molecular physics within the domain of mathematics. **Clausius**, who was among the earliest to discuss the subject of heat from a mathematical point of view. **Clebsch**, who has discussed the elasticity of solid bodies. **Faraday**, see p. 441. **Foucault**, see p. 442.

Green, see p. 443. **Helmholtz**, who is in the front rank of all departments of mathematical physics. **Lamé**, see p. 442. **Clerk Maxwell**, see pp. 443–5. **MacCullagh**, who wrote on physical optics. **F. Neumann**, who has written on elasticity and light. **Poincaré**, who has discussed the form assumed by a mass of fluid under its own attraction. **Rankine**, whose discoveries in thermodynamics and hydromechanics will be found in the collected edition of his works issued in London in 1881. **Lord Rayleigh**, who has written the standard work on sound, published at Cambridge in 1877. **Saint-Venant**, whose researches on torsion are well known. **Stokes**, most of whose papers are on hydromechanics or optics or allied subjects; these memoirs have been recently collected and published by the university of Cambridge. **Sir William Thomson**, to whom the compliment of publishing his collected papers has also been recently offered by the university of Cambridge, has enriched every department of physics by his researches; but perhaps his papers on electricity and hydrodynamics may be singled out as specially characteristic of his genius. **Weber**, whose chief work was in connection with electrodynamics. **Wiedemann**, who is the author of an admirable text-book on electricity and the allied subjects, 4 vols., 1882–1885. I add a few notes on one or two of these writers, but I repeat again that the subject of physics lies outside the limits of this book, and the above list of writers does not in any way profess to be complete or exhaustive.

Michael **Faraday** was born at Newington on Sept. 22, 1791 and died at Hampton Court on Aug. 25, 1867. Faraday is the most original and brilliant experimental physicist of this century; but though he had no knowledge of the higher parts of mathematical analysis, he was able to deduce many results by general reasoning from fundamental principles, and no modern writer has shown an equal skill in disentangling those principles from the symbols in which they are usually expressed. He was the son of a blacksmith, and was apprenticed to a book-binder; while working at his trade he educated himself; his abilities having attracted the attention of Davy,

he was made an assistant at the Royal Institution in London, and ultimately became professor there. His earliest discoveries were on chemistry. His experiments on the induction of electric currents lasted from 1821 to 1831, and were crowned with complete success. In the next ten years he established the law of definite electrolytic action of a current, and amongst other things determined the specific inductive capacities of various substances. In 1845 he established the effect of magnetism on polarized light, and discovered the phenomena of diamagnetism. His life has been written by Tyndall (2nd ed. 1870), Bence Jones (1870), and Dr Gladstone (1872); these biographies may be consulted for further details.

Gabriel **Lamé**, born at Tours on July 22, 1795 and died at Paris in 1870, was educated at the Polytechnic school, and on leaving that was employed for some years in the engineering service of the Russian government. On his return to France in 1832 he was appointed professor at his old school, and took an active part in promoting the construction of railways in France. His best-known works are on various branches of mathematical physics. The most prominent of these are his course on physics, 1836; his treatise on elasticity, 1852; his work on functions, 1857; an essay on curvilinear coordinates, 1859; and lastly his theory of heat, 1861. He also wrote several memoirs on different points in the theory of numbers.

Jean Bernard Léon **Foucault**, born at Paris on Sept. 18, 1819 and died there of paralysis on Feb. 11, 1868, is among the most eminent of modern French physicists. He was the son of a well-known publisher, and was educated at home until he entered the Paris hospitals. He soon abandoned medicine for physics. His scientific papers were nearly all contributed to the Comptes Rendus. His chief memoirs are on the practicability of photography, 1840; on the electric lamp, 1849; on the determination of the velocity of light, in 1850, but repeated with improvements in 1862; on his demonstration of the diurnal motion of the earth by means of the rotation of the plane of oscillation of a simple pendulum, 1851; on his in-

vention of the gyroscope, 1852; on the rotation of a copper disc between the poles of a magnet, 1855; and on his polarizer, 1857. For additional details see La vie et les travaux de Léon Foucault by Lissajous, Paris, 1875.

I come now to two writers who treated the subject from a more strictly mathematical point of view, namely Green and Clerk Maxwell.

George Green, born near Nottingham in 1793 and died at Cambridge in 1841, was one of the most remarkable geniuses of this century, though he published but little. Although self-educated he contrived to obtain copies of the chief mathematical works of his time. A paper of his written in 1827 was published by subscription in the following year: the term potential was here first introduced, its leading properties proved, and the results applied to magnetism and electricity. In 1832 and 1833 papers on the equilibrium of fluids and on attractions both in space of n dimensions were presented to the Cambridge Philosophical Society, and in the latter year one on the motion of a fluid agitated by the vibrations of a solid ellipsoid was read before the Royal Society of Edinburgh. In 1833 he entered at Caius College, Cambridge: he took his degree in 1837, and in 1839 got a fellowship. Directly after taking his degree he threw himself into original work, and produced in 1837 his paper on the motion of waves in a canal, and on the reflexion and refraction of sound and light. In the latter the geometrical laws of sound and light are deduced by the principle of energy from the undulatory theory, the phenomenon of total reflexion is explained physically, and certain properties of the vibrating medium are deduced. In 1839, he read a paper on the propagation of light in any crystalline medium. All the papers last-named are printed in the Camb. Phil. Trans. for 1839. A collected edition of his works was published at Cambridge in 1871.

James Clerk Maxwell, born at Edinburgh on June 13, 1831 and died at Cambridge on Nov. 5, 1879, was educated at Edinburgh and Trinity College, Cambridge, of which latter

society he was a fellow. He was successively professor at Aberdeen from 1856 to 1860, and at King's College, London, from 1860 to 1868; in 1871 he was appointed to the Cavendish chair of physics at Cambridge. His numerous memoirs prove him to have been a mathematical physicist of the first rank. His earliest paper was written when only fourteen on a mechanical method of tracing cartesian ovals, and was sent to the Royal Society of Edinburgh. His next paper written three years later was on the theory of rolling curves, and was immediately followed by another on the equilibrium of elastic solids. At Cambridge in 1854 he read papers on the transformation of surfaces by bending, and on Faraday's lines of force. These were followed in 1859 by the essay on the stability of Saturn's rings, and various articles on colour. But brilliant though these memoirs are they are eclipsed by his work on electricity and the kinetic theory of gases.

His *Electricity and Magnetism*, in which the results of various papers are embodied, was issued in 1873, and has revolutionized the treatment of the subject. Poisson and Gauss had shown how electrostatics might be treated as the effects of attractions and repulsions between imponderable particles; while Sir William Thomson in 1846 had shown that the effects might also and with more probability be supposed analogous to a flow of heat from various sources of electricity properly distributed. In electro-dynamics the only hypothesis then current was the exceedingly complicated one proposed by Weber in which the attraction between electric particles depended on their relative motion and position. Maxwell rejected all these hypotheses and proposed to regard all electric and magnetic phenomena as stresses and motions of a material medium; and these, by the aid of generalized coordinates, he was able to express in mathematical language. He concluded by showing that if the medium were the same as the so-called luminiferous ether, the velocity of light would be equal to the ratio of the electro-magnetic and electrostatic units. This appears to be the case though these

units have not yet been determined with sufficient precision to enable us to speak definitely on the subject.

Hardly less eventful, though less complete, was his work on the kinetic theory of gases. The theory had been established by the labours of Joule in England and Clausius in Germany; but Maxwell reduced it to a branch of mathematics. He was engaged on this subject at the time of his death and his two last papers were on it. It has been the subject of some recent papers by Boltzmann.

Amongst the other contributions of Maxwell to science are The electrical researches of Cavendish issued in 1879 (see p. 399), his Theory of heat published in 1871, and his elementary text-book on Matter and Motion: to which I may add the four articles entitled Atom, Capillary attraction, Constitution of bodies, and Diffusion which he contributed to the ninth edition of the Encyclopaedia Britannica. For further details his life by Campbell may be consulted. His collected works are being edited by Prof. Niven and will shortly be published by the university of Cambridge.

INDEX.

Abacus; description of, 116–9.
— ref. to, 6. 123. 128. 168. 169.
Abd-al-gehi, 155.
Abel, 416–7.
— ref. to, 365. 395. 416. 417. 418. 423. 435. 437.
Abelard, 180.
Abelian functions, chap. XIX, sect. 1.
Aberration; astronomical, 300.
Abul-Wafa, 155.
Academy; the French, 350.
— the Berlin, 340.
Adalbert of Rheims, 127.
Adams, 412.
— ref. to, 286. 300. 308. 430. 440.
Addition; processes of, 171.
— symbols for, 9. 96. 98. 148. 174. 185. 186. 189. 191. 192. 208. 212. 213.
Adelhard of Bath, 158.
Ahmes, 4–10. ref. to, 84. 95.
Albategni, 154.
Alberi on Galileo, 222.
Albuzjani, 155.
Alcuin, 124–5.
Alembert d'. See d'Alembert.
Alexander the Great, 43. 46. 47.
Alexandria; the university of, 46. 47. 85. 86. 105. 109.
Alexandrian library; the, 47. 109.
— schools; the, chapters IV. V.
— symbols for numbers, 120.
Alfarabius, 159.
Alfred the Great, 107.
Algebra. Treated geometrically by Euclid and his school, 58. 95. Development of rhetorical and syncopated algebra in the fourth century after Christ, 94–100. Discussed rhetorically by nearly all the Hindoo and Arab mathematicians, chapters IX. X. Introduction of syncopated algebra by Bhaskara, 145; Regiomontanus, 184; and Pacioli, 189; and of symbolic algebra by Vieta, 204; see chapter XII. Developed by (amongst others) Descartes, 244; Wallis, 259; Newton, 334; and Euler, 367. 369. Recent extensions of, chapter XIX, section 3.
Algebra; definitions of, 160.
— division into rhetorical, syncopated, and symbolic, 95. 96.
— origin of term, 151.
— histories of, ix. x. 250.
Algebraical equations. See simple equations, quadratic equations, &c.
— problems; earliest collection of, 94.
— symbols; origin of, 212–5.
— theorem; the earliest known, 88.
Algorism, 158. 159. 169. 170. 185.
Alhazen, 155. 159.
Alhossein, 154.
Alkarismi, 150–3.
— ref. to, 158. 159. 160. 169. 200.
Alkarki, 154.
Alkhodjandi, 153.
Al-Khwārizimī. See Alkarismi.
Allman on Greek geometry, ix. 22. 25. 27. 28. 32. 35. 40. 43. 44.
Almagest; the, 90–1.
— ref. to, 74. 79. 103. 155. 158. 159. 165. 180.
Al Mamun, the caliph, 140. 151.
Al Mansur, the caliph, 141.

Alphonso's tables, 161.
Al Raschid, the caliph, 140.
Amasis of Egypt, 15.
Ampère, 405.
Analysis; algebraical, 97.
— geometrical, 40.
Analytical geometry. See Geometry.
Anaxagoras of Clazomenæ, 31–2.
Anaximander, 16. 17. 18.
Anchor ring; the, 42. 78.
Anderson, 207.
Angle; origin of sexagesimal division of, 6. 215.
— trisection of, 32. 34. 209. 350.
Angular coefficient, 270.
Antioch; Greek school at, 140.
Antipho, 33.
Apollonius, 70–5.
— ref. to, 83. 103. 104. 141. 153. 163. 202. 206. 209. 242. 260. 268. 300. 305. 350.
— and Archimedes; contrast between geometry of, 75.
Appell, 412. ref. to, 430.
Apse of lunar orbit; motion of, 315. 319. 352.
Arab works introduced into Europe, chap. x.
Arabic numerals; origin of, 147.
Arabs; mathematics of the, chap. IX.
Arago, 406–7.
— ref. to, ix. 83. 371. 405. 421. 439.
Aratus, 43. 79.
Arbogaste, 372.
Archimedes; life and works of (see table of contents), 59–70.
— inventions of, 59. 60.
— geometrical principles assumed by, 69.
— ref. to, 57. 72. 74. 78. 83. 94. 104. 141. 153. 155. 157. 163. 202. 216. 228. 256. 268. 345.
— and Apollonius; contrast between geometry of, 75.
Archytas, 27–9.
— ref. to, 26. 34. 39. 41.
Areas; equable, description of, 226. 295. 298. 313.
Argyrus, 112.
Aristæus, 34. 44. 52. 71. 350.
Aristarchus, 57–8. ref. to, 202.
Aristotle, 45.
Aristotle, ref. to, 13. 26. 107. 140. 202.
Aristoxenus, 21.
Arithmetic. Pre-hellenic, 3–6. Pythagorean, 25–7. Practical Greek, 53. 120. 121. Theory of, treated geometrically by most of the Greek mathematicians to the end of the first Alexandrian school, 52–5; and treated empirically (Boethian arithmetic) by most of the Greek and European mathematicians of the first fourteen centuries after Christ, 88. 121. 168. Algoristic arithmetic invented by the Hindoos and Arabs, 143. 146. 153; and used since the fourteenth century in Europe, 161. 169. 170. Development of European arithmetic, chapter XI.
— histories of, ix. x. 114. 168.
Arithmetical problems, 56. 66.
— progressions, 64. 78. 146.
— triangle of Pascal, 207. 252.
'Αριθμητική; signification of, 53.
Arneth, ix.
Aronhold, 412. 432.
Arya-Bhatta, 142–8.
— ref. to, 144. 149. 155.
Arzachel, 158.
Assumption; rule of false, 95. 140. 162. 186. 187.
Assurance, life, 275.
Astronomy. Descriptive astronomy outside range of work, vi. Early Greek theories of, 16. 22. 28. 31. 43. 56. 57. 69. 74. Scientific astronomy founded by Hipparchus, 79; and developed by Ptolemy in the Almagest, 90. Studied by Hindoos and Arabs, 154. 158. Modern theory of, created by Copernicus, 190; Galileo, 220; and Kepler, 226. Physical astronomy created by Newton, chapter XVI. Development of, by (amongst others) Clairaut, 352; Lagrange, 376. 378; Laplace, 385–7; and in recent times by Gauss and others, chapter XIX.
— histories of, ix. x.

Asymptotes; theory of, 322.
Athenian school; the later, 101.
Athens; the school of, chap. III.
Atomistic school; the, 30.
Attalus, 71.
Attic symbols for numbers, 120.
Attractions; theory of, 266-7, 295-9, 317, 319, 362, 370, 381, 392, 405, 406, 423, 433.
history of, x.
August; edition of Euclid, 51.
Autolycus, 56.
Avery's steam engine, 64.

Babbage, 409-10. ref. to, 381.
Bachelor of arts; degree of, 135.
Bachet, 196-7. ref. to, 261, 377.
Bacon, Francis, 222-3. ref. to, 266.
Bacon, Roger, 163-5.
ref. to, 156, 161.
Baillet; life of Descartes, 236.
Baldi; life of Arzachel, 160.
Ball, Sir Robert, 412. ref. to, 403.
Balsham, bishop of Ely, 135.
Barlaam, 116.
Barlow on theory of numbers, 377.
Barometer; invention of, 250, 370.
Barral on Arago, 407.
Barrow, 266-70.
ref. to, 55, 88, 249, 258, 265, 281, 287, 290, 301, 324, 327, 342.
Beaune, de, 275.
Becket, Thomas à, Archbishop, 130.
Bede on finger symbolism, 112.
Beeckmann, 238.
Beltrami, 413, 436.
Benedictine monasteries; schools at the, 123.
Ben Ezra, 158-9. ref. to, 161.
Berlin Academy; creation of, 310.
Bernelinus, 128.
Bernhardy; ed. of Eratosthenes, 77.
Bernoulli, Daniel, 317.
ref. to, 318, 319, 365.
Bernoulli, Jacob, 315. ref. to, 351.
Bernoulli, John, 315-6.
ref. to, 215, 306, 310, 323, 330, 342, 346, 348, 355, 361, 366, 367, 368.
Bernoullis, the younger, 317.
Bertrand, 412. ref. to, 400, 430.
Berulle, de, Cardinal, 238.
Bessel, 398-9.
Beta function, 308, 394.
Betti, 412, 432.
Bézout, 372.
Bhaskara, 145-9. ref. to, 155.
Binomial equations, 377, 382, 423.
Binomial theorem; discovery of, 253, 284, 291, 324, 369.
Biot, 406. ref. to, 2, 307, 387.
Biot and Lefort on Leibnitz, 329, 330, 332.
Biquadratic reciprocity, 394.
Biquadratic residues, 422.
Biquadric equations, 199, 201, 208.
Birch; life of Boyle, 270.
Bjerknes on Abel, 417.
Bija ganita; the, 145, 148-9.
Böckh; ref. to, 4.
Bode's law, 386.
Boethius, 106-8.
— ref. to, 88, 125, 126, 138, 158, 164, 168, 171.
Bologna; university of, 129, 130.
Boltzmann, 412. ref. to, 440, 445.
Bombelli, 203.
— ref. to, 199, 201, 204, 205, 214, 215.
Bonnet, 159.
Boncompagni; ref. to, x, 151, 158, 159, 162, 185.
Boniface VIII. of Rome, 134.
Book-keeping by double entry, 170, 187.
Bosch, 430.
Booth, 412. ref. to, 434.
Borchardt, 412. ref. to, 417, 424, 431.
Borrel, 202.
Bossut; ref. to, ix, 359.
Bougainville, 348.
Bouquet, 412, 420.
Boyle, 270.
Brachistochrone; problem of the, 305, 313, 316, 348, 368.
Brackets; introduction of, 214.
Bradley, 369.
Brahmagupta, 143-4.
— ref. to, 145, 147, 149, 150, 151, 155, 170, 184, 277.
Brassine; edition of Fermat, 261.
Breitschwert; life of Kepler, 224.
Bretschneider; ref. to, ix, 38, 52.
Brewster; works by, 218, 283, 321,

Briggs, 177–7, 309.
Brinkley, 412, 432.
Briot, 112. ref. to, 420.
Brouncker, 277 n. ref. to, 113, 249.
Budan, 403.
Buffon on Archimedes' burning mirrors, 60.
Bulmer; ref. to, 402.
Byzantine school; the, chap. vi.

Calculus. See Differential, Functional, Integral, Variations, &c.
Cambridge; the university of, [illegible]
Campanus, 165. ref. to, [illegible]
Campbell, 300.
Cantor; ref. to, [illegible]
66, 77, 96, 105, 111, [illegible]
Capet, Hugh, of France, 127.
Capillarity, [illegible]
Caravi, 260, 261.
Cardan, 197–208.
ref. to, 190, 213 [illegible]
259, 376.
Carcavi; edition of Descartes, 258.
Carnot, Lazare, 397–9.
ref. to, 91, 383, 4[illegible]
Carnot, Sadi, 402.
Cassiodorus, 109.
Castillon on Pappus' problem, [illegible]
Cataldi, 210. ref. to, [illegible]
Catenary, 273, 331, [illegible]
Cathedral schools, [illegible]
Cauchy, 433–9.
ref. to, 267, [illegible]
430.
Cavalieri, 257–9.
ref. to, 225, [illegible]
Cavendish, [illegible] ref. to, [illegible]
Cayley, 442.
ref. to, 410, [illegible]
Censura, [illegible]
Centroid, 155.
Centre of mass, [illegible]
Centrifugal force, [illegible]
Chalcidius, [illegible]
Charles, [illegible]

Charles I. of England, 294.
Charles II. of England, 294.
Charles the Great; schools of, 124–5.
Chasles, 443.
ref. to, 18, 55, 75, 295, 381.
[illegible], ref. to, 106.
[illegible], 270, 333, 404.
[illegible] of mathematics, [illegible]
[illegible], 253.
[illegible] to the Lord opposed to [illegible], 101, 104, 105, 109.
[illegible], 14.
[illegible], discovery of, 226, 238.
Circle, quadrature of, or squaring the, [illegible]
[illegible]
[illegible]
[illegible]
[illegible]
[illegible]
[illegible]
[illegible]
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[illegible]
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[illegible]
[illegible]
[illegible]
[illegible]

48; especially by Euclid, 55; and Apollonius, 71–2. Interest in, revived by writings of Kepler and Desargues, chapter XIII, section 2; and subsequently by Pascal, 253; Newton, 314; and Maclaurin, 350. Treatment of by modern geometry, chapter XVIII, section 3; chapter XIX, section 4.
— histories of, ix. x.
Conicoids, 308. 377. 390.
Conoids and spheroids, 65.
Conon of Alexandria, 58. 59. 64.
Conservation of energy, 313. 397. 413.
Constantine VII, the emperor, 111.
Conti, 331.
Continued fractions, 210. 278. 382. 389.
Continuity; principle of, 225. 237. 334. 341. 398.
Convergency, 334. 367. 369.
Commercium Epistolicum; the, of Newton, 329. 330.
Copernicus, 190.
— ref. to, 60. 91. 201. 208. 220. 221.
Corpuscular theory of light, 291. See Optics (physical).
Coss, 188.
Cosecant; origin of term, 215.
Cosine, 190, 212, 215.
Cos x; series for, 278.
Cos⁻¹ x; series for, 278.
Cossic art; the, 188.
Cotangent; origin of term, 215.
Cotangents; table of, 155.
Cotes, 358.
— ref. to, 301. 309. 316. 359. 380.
Cousinery, 377.
Courçon, Cardinal de, 131.
Courier on Laplace, 391.
Cousin; edition of Descartes, 240.
Cramer, 333. ref. to, 93. 316.
Cremona, 412. ref. to, 395. 433.
Crofton, 412. 430.
Ctesibus, 81.
Cube; duplication of a, 28. 34. 39. 44. 74. 88. 299.
— origin of problem, 38. 75.
Cubic curves, 321–3.
Cubic equations, 65. 99. 203. 208. 209.
Cubic equations, solutions of, 153. 154. 193. 194. 198. 199. 200. 208.
Cubic reciprocity, 394. 424.
Curvature; lines of, 396.
Curve of quickest descent, 305. 343. 346. 348. 368.
Curves; areas of. See Quadrature.
Curves; classification of, 242–3. 321–2.
Curves of the third degree, 321–3.
Curves; rectification of, 258. 280. 324.
Curves; tortuous, 352. 368.
Cycloid; the, 255. 258. 271. 272. 275. 276. 316. 349. 351.
Cyzicus; the school of, chapter III.

D'Alembert, 353–5.
— ref. to, 256. 313. 318. 347. 357. 365. 369. 374. 378. 383. 384.
Dalton, 400–1.
Damascius, 104.
Damascus; Greek school at, 140.
Darboux, 412. ref. to, x. 435. 436.
Darwin, 413. ref. to, 440.
Davy, 441.
De Beaune, 275.
De Berulle, Cardinal, 238.
Decimal; introduction of decimal point or bar, 176–7.
Decimal fractions; notation for, 176. 317.
Decimal numeration in language, 65. 115.
— writing, 65. 73–4. 147. 150. 159. 160. 161. 170.
De Courçon, Cardinal, 131.
Dedekind, 413.
— ref. to, 419. 424. 428. 436.
Deduction; characteristic of Greek geometry, 9. 14.
Defective numbers, 27.
De Gua, 340.
De Kempten, 115.
De Lahire, 351. ref. to, 276.
Delambre; ref. to, ix. 79. 80. 91. 149. 208. 385.
Delaunay, 413. ref. to, 439.
De Méré, 253.
Democritus, 30.
De Moivre, 358–9. ref. to, 371.
De Montmort, 349.

De Morgan, 430–1.
— ref. to, 48, 50, 52, 53, 55, 89, 91, 102, 168, 169, 185, 189, 255, 278, 329, 380, 382, 387, 388.
Denifle on mediæval universities, 128.
De Rohan, the duke, 205.
Desargues, 226–7.
— ref. to, 225, 230, 237, 253, 351, 395.
Descartes, 230–40.
— ref. to, 51, 204, 211, 213, 215, 223, 227, 229, 233, 255, 256, 258, 260, 264, 266, 271, 272, 275, 277, 284, 289, 305, 345.
— rule of signs of, 199, 244, 349.
De Sluze, 280. ref. to, 279.
Desmaze on Ramus, 203.
De Sorbonne, 131.
Destouches, 353.
Determinants, 323, 344, 373, 377, 389, 418, 423, 432, 438.
Dewulf; ed. of Cremona, 435.
Differences; calculus of finite, 349, 372, 378, 389, 430.
Differential calculus, 233, 293, 328, 340, 341, 342, 345, 346, 348, 361, 368, 408, 409, 431.
Differential equations, 372, 377, 383, 389, 397, 418, 430, 431, 432, 436.
— solution of a partial, of second order, 354.
Differential triangle; the, 269, 270.
Diffraction, 301, 321.
Dinocrates, 46.
Diocles, 78. ref. to, 85.
Dionysius, 29.
Dionysodorus, 85.
Diophantus, 96–103.
— ref. to, 26, 95, 111, 141, 154, 181, 197, 202, 203, 261, 262, 291, 383.
Directrix in conics discovered by Pappus, 92, 93.
Dirichlet, 424. ref. to, 418.
Disturbing forces, 314, 317, 376.
Ditton, 356.
Division; processes of, 172–4.
— symbols for, 154, 210, 214.
Doctor; degree of, 131.
Dodecahedron; discovery of, 21.
Dodson, 275.
Dositheus, 57, 62, 64, 65.
Double entry; book keeping by, 170, 187.
Ducks and drakes, 356.
Duhring; hist. of mechanics, ix.
Duillier, 329.
Dupin on Monge, 395.
Duplication of a cube. See Cube.
D'Orville; ed. of Aristarchus, 57.
Dürer, 161.
Durham, university of, 190.
Dynamics. See Mechanics.

Earth; density of, 399.
radius of, 65.
Eclipse foretold by Thales, 16.
Edessa; Greek school at, 110.
Edleston; correspondence of Newton and Cotes, 303.
Edward I. of England, 165.
Edward III. of England, 186.
Edward VI. of England, 191.
Egbert, Archbishop, 133.
Egyptian mathematics, 4–10, 84, 110.
Eisenlohr on the Rhind papyrus, 4, 8.
Eisenstein, 424.
ref. to, 416, 425, 426, 427, 428, 431.
Elastic string; tension of, 370.
Elastica; the, 315.
Elasticity; theory of, 410, 441, 443.
Eleatic school; the, 29–30.
Electricity, 325, 404, 405, 441, 442, 443, 444.
Elements of Euclid, 49–52.
— ref. to, 65, 73, 76, 78, 79, 103, 104, 107, 141, 153, 155, 161, 165, 166, 202, 240, 250, 264, 393. See also Euclid.
Elimination; theory of, 372, 377.
Ellipse; rectification of, 292, 363.
Elliptic functions, 359, 391, 395; chapter xix, section 1; 424, 430.
Elliptic orbits of planets, 158, 230.
Ellis, Spedding and Heath; on Francis Bacon, 221.
Elizabeth, Queen, 211.
Emesa; Greek school at, 140.

Emission theory of light, 291. See Optics (physical).
Energy, 312, 318.
  conservation of, 318, 397.
Eneström; math. Journ. of, x.
Engelmann; edition of Bessel, 430.
Envelopes; determination of, 272, 290, 342.
Epicycles, 89, 91.
Equality; symbols for, 9, 98, 174, 180, 191, 214.
  ; meanings of, 191, 205, 214.
Equations. See Simple equations, Quadratic equations, &c.
Equiangular spiral; the, 345.
Eratosthenes, 76-7. ref. to, 39, 70.
Ersch and Gruber on Descartes, 236.
Ernes, 255.
Ether; the luminiferous, 378, 444.
Euclid, 48-56.
  — ref. to, 33, 39, 61, 83, 91, 141, 153, 155, 157, 183, 212, 291, 378. See also Elements of Euclid.
Euc. ix. 19; Ptolemy's proof of, 89.
Euc. i. 5, ref. to, 14, 161.
— i. 12, ref. to, 29.
— i. 13, ref. to, 23.
— i. 15, ref. to, 14.
— i. 23, ref. to, 29.
— i. 26, ref. to, 14.
— i. 29, ref. to, 23.
— i. 32, ref. to, 15, 23, 250.
— i. 44, ref. to, 25.
— i. 45, ref. to, 25.
— i. 47, ref. to, 3, 7, 23, 26, 144.
— i. 48, ref. to, 7, 23.
— ii. 2, ref. to, 24.
— ii. 3, ref. to, 97.
— ii. 8, ref. to, 97.
— ii. 11, ref. to, 41, 53.
— ii. 14, ref. to, 25, 53.
— iii. 16, ref. to, 28.
— iii. 31, ref. to, 15, 36.
— iii. 35, ref. to, 28.
— vi. 2, ref. to, 14.
— vi. 4, ref. to, 14, 24.
— vi. 17, ref. to, 24.
— vi. 25, ref. to, 25.
— vi. 28, ref. to, 53, 94.
— vi. 29, ref. to, 53, 94.
Euc. vi. D. ref. to, 81.
— x. 1. ref. to, 42.
— x. 9. ref. to, 44, 55.
— x. 117. ref. to, 55.
— xi. 19. ref. to, 28.
— xii. 2. ref. to, 36, 42.
— xii. 7. ref. to, 42.
— xii. 10. ref. to, 42.
— xiii. 1–5. ref. to, 41, 52.
— xiii. 6–12. ref. to, 52.
— xiii. 13–18. ref. to, 52.
— xiv. ref. to, 78.
— xv. ref. to, 104.
Eudemus of Athens, 12, 15, 17, 39.
Eudemus of Pergamum, 71.
Eudoxus, 40-3.
— ref. to, 34, 43, 49, 50, 54.
Euler, 360-71.
— ref. to, 19, 93, 113, 172, 199, 212, 215, 261, 262, 267, 302, 318, 333, 335, 347, 355, 363, 365, 373, 374, 375, 378, 380, 396, 406, 425.
Eurytas of Metapontum, 39.
Eutocius, 104. ref. to, 71, 121.
Evolutes, 272.
Excentrics, 89, 91.
Excessive numbers, 27.
Exchequer division; court of, 168.
Exhaustions; method of, 42, 76, 247, 275.
Expansion in series, 367.
Exponential calculus, 346.
Exponential series, 344.
Exponents, 203, 214, 215, 217, 244, 256, 324.

Fabricius, 29.
Fagnano, 350.
False assumption; rule of, 95, 146, 162, 186, 187.
Faraday, 441–2. ref. to, 407, 444.
Favaro; edition of Galileo, 222.
Faye on the nebular hypothesis, 386.
Fermat, 260–7.
— ref. to, 113, 203, 243, 245, 250, 251, 253, 255, 256, 271, 273, 277, 369, 374, 377, 383.
—'s equation, 262 (*i*), 153, 377.
—'s theorem, 261 (*a*), 382.
Ferrari, 201. ref. to, 198, 199, 208.
Ferro, 193.

Fibonacci, 159–62.
Figurate numbers, 252.
Finck, 215.
Finger symbolism, 106. 112.
Finite differences, 349. 372. 378. 389. 430.
Fiori, 193. 198.
Fire engine invented by Hero, 83.
Fischer on Descartes, 236.
Five; origin of word, 114.
Fizeau; ref. to, 407.
Flamsteed, 302.
— ref. to, 300. 303. 390.
Fluctuating functions, 430.
Fluents; Newton on, 325–8.
Fluxional calculus; 284. 285. 325–8. 361.
— ref. to, 232. 233. 301.
— controversy, 328–33.
Fluxions; Newton on, 325–8.
Focus of a conic, 72. 92. 225.
Focus; orbit in a conic about a, 295–8. 314.
Fontana. See Tartaglia.
Forces; measurement of, 311.
Forces; parallelogram of, 45. 217. 312.
Forsyth, 415.
Foucault, 442–3. ref. to, 407.
Fourier, 401–2. ref. to, 365.
Fractions; treatment of, 5. 6. 88.
— continued, 210. 278. 382. 389.
Frederick II. of Germany, 162–3.
— ref. to, 161.
Frederick the Great of Prussia, 351. 366. 375. 379.
French Academy, 250.
Frénicle, 276. ref. to, 264.
Fresnel, 405–6. ref. to, 407.
Friedlein; ref. to, 81. 96. 104. 107. 114.
Frisch; edition of Kepler, 226.
Frisi; life of Cavalieri, 247.
Froben; edition of Alcuin, 124.
Frobenius, 419. 436.
Fuchs, 413. ref. to, 436.
Functional equations, 372.
Functions; calculus of, 431. 436. 442.
Fuss, 306. ref. to, 93.

Galande; the, 270.
Gale; edition of Archytas, 29.
Galen, 140.
Galileo, 218–22.
— ref. to, 190. 216. 225. 228. 236. 237. 255. 343. 350.
Galley system of division, 172–4.
Galois, 413. ref. to, 431.
Gamaches, 319.
Gamma function, 368. 394. 422.
Garth, 171.
Gassendi on Regiomontanus, 170.
Gauss, 420–3.
— ref. to, 199. 307. 365. 382. 388. 390. 393. 394. 395. 418. 419. 420. 424. 425. 427. 432. 437. 438. 441.
— character of his analysis, 423.
Geber ibn Aphla, 158.
Geiser and Schröder; ed. of Steiner, 433.
Gelder on Theon of Smyrna, 88.
Gelon of Syracuse, 65.
Geminus, 12.
Generalized coordinates, 378.
Geodesics, 316, 368.
Geometrical curves, 242.
Geometrical progressions, 54. 63. 146.
Geometry. Egyptian, 7. 8. Greek, discussed or used by nearly all the mathematicians considered in the first period, chapters II–V; also by Newton, 309. 314; and his school, chapter XVII, section 3; see also Conic sections. Arab and Mediæval; founded on Greek works, chapters VIII, IX, X. Geometry of the renaissance; characterized by a free use of algebra and trigonometry, chapters XII, XIII.
Geometry, Analytical, 241; discussed or used by nearly all the mathematicians considered in the third period, chapters XIV–XIX.
— Modern, chapter XIII, section 2; chapter XVIII, section 3; chapter XIX, section 4.
— histories of, ix. x. 12. 92. 104.
— origin of, 6. 7.
George I. of England, 340.
Gerard, 159. ref. to, 158. 161.
Gerbert (Sylvester II.), 126–8.

Gerhardt, 111. 341.
Ghetaldi, 73.
Giraldus Cambrensis, 133.
Girard, 241. 316.
Gladstone; life of Faraday, 412.
Glaisher; ref. to, 327. 415. 429.
Gloomed school at Cambridge, 135.
Gnomon or style; the, 17.
Gnomons or odd numbers, 25.
Godfrey, 315.
Golden section; the, 41. 53.
Göpel, 413. ref. to, 410.
Gordan, 413. 432.
Gothals on Stevinus, 316.
Gow; ref. to, ix. 1. 70.
Grant; hist. of astronomy, ix.
Grassmann, 50. 413. 433.
Graves, life of Hamilton, 430.
Gravity; centres of, 67. 181. 258. 266.
Gravity; law of, 295-7. 307. 298.
symbol for, 310.
value of, 270. 310. 352.
Greatest common measure; Euclid on, 54.
Green, 413. ref. to, 391.
Greenwood; edition of Hero's Pneumatica, 83.
Gregory, David, 350.
ref. to, 290. 292. 301. 303. 351.
Gregory, James, 279.
ref. to, 249. 350. 359.
Gregory XIII. of Rome, 196.
Gross on Chinese dups, 250.
Grosseteste, 101.
Grube on Dirichlet, 421.
Gua, de, 343.
Guhrauer; life of Leibnitz, 330.
Guldinus, 223. ref. to, 325.
Günther; hist. of mathematics, ix.
Günther, 176. 215.
Gutenäcker; ed. of Archimedes, 70.

Hadley; ref. to, 365.
Halley, 360.
ref. to, 72. 73. 75. 87. 279. 280. 287. 291. 296. 307. 300. 302. 350. 379.
Halphen, 414. ref. to, 420. 425. 436.
Hamilton, Sir Wm., 429.
ref. to, 103. 378.
Hamilton, Vacc., 110.
Hankel; ref. to, ix. 55. 95. 114. 190. 192.
Hansen, 413. ref. to, 439.
Harmonic series, 27. 288.
Harmonics; spherical, 385. 392.
Harriot, 240-1.
— ref. to, 204. 213. 214.
Hawksbee, on capillary attraction, 380.
Heat; theory of, 295. 401. 402. 404. 410. 412. 445.
Heath on Diophantus, 96.
Heiberg; ref. to, 48. 59. 70. 72.
Heilbronner; hist. of math. ix.
Helix, 276.
Heller; history of physics, ix.
Hellner; edition of Borchardt, 431.
Helmholtz (von), 413.
— ref. to, 234. 433. 440.
Henry II. of England, 130.
Henry III. of England, 130. 135. 136.
Henry VIII. of England, 184.
Henry IV. of France, 206.
Henry, C.; ref. to, 93. 101. 212. 237. 281.
Heracleides, 71.
Herigone, 215.
Hermite, 413. ref. to, 419. 428. 431.
Hero of Alexandria, 81-4.
— ref. to, 94. 121. 202.
Hero of Constantinople, 110-11.
Herschel, Sir John, 409-10.
— ref. to, 393. 408.
Herschel, Sir William, 410.
Hesse, 413. ref. to, 435.
Hiero of Syracuse, 59. 68.
Hill, 413. ref. to, 440.
Hiller; ed. of Theon of Smyrna, 88.
Hindoo mathematics. Chapter ix, section 2.
Hipparchus, 79-81.
— ref. to, 81. 96. 155.
Hippasus, 21.
Hippias, 32-3.
Hippocrates of Chios, 35-30.
— ref. to, 44. 48.
Hippocrates of Cos, 35. 140.
Hire, de la, 351. ref. to, 270.
History of mathematics; authorities for, ix. x.
— divided into three periods, 1.

Hodograph, 430.
Hoche; edition of Nicomachus, 88.
Hochheim; edition of Alkarki, 151.
Hoefer; hist. of mathematics, ix.
Hoffmann on Euc. I. 47, 25.
Homogeneity; importance of, 200.
Homology, 227.
Honein ibn Ishak, 140.
Hooke, 279-80.
— ref. to, 278, 279, 290, 291, 295, 298, 301, 380, 406.
Horsley; edition of Newton, 307.
Hospital, l', 348.
Huber; life of Lambert, 371.
Hudde, 276.
Hugens. See Huygens.
Hultsch; ref. to, 81, 83, 93.
Humboldt, 421.
Hutton; ref. to, ix. 197.
Hutton Shaw and Pearson, ix.
Huygens, 270-4.
— ref. to, 231, 251, 259, 275, 276, 279, 280, 290, 291, 298, 310, 330, 406.
Huyghens. See Huygens.
Hydrodynamics; created by Newton, 318; developed by (amongst others) Maclaurin, 362; Euler, 370; and Laplace, 387.
Hydrostatics; treatment of by Archimedes, 68, 69; by Stevinus, 218; by Galileo, 220; by Pascal, 250-1; by Newton, 319; by Euler, 370.
Hypatia, 108.
Hyperbolic functions, 371.
Hyperboloid of one sheet, generators of, 279.
Hypergeometric series, 422.
Hypsicles, 78.

Iamblichus; ref. to, 19. 110.
Imaginary quantities, 199. 263. 334. 359. 398.
Inception to degrees, 181.
Incommensurables; Euclid's treatment of, 52. 54. 55.
Indestructibility of matter and energy, 318.
Indeterminate coefficients, 314.
— forms, 348.
Indian mathematics, chapter ix, section 2.
Indian numerals; origin of, 147.
Indices, 203, 211, 214, 215, 217, 244, 256, 321.
Indivisibles; method of, 235, 262, 247, 249, 254.
Induction; characteristic of Egyptian geometry, 9.
— [illegible]
Inertia of a body, 311.
Infinite series, difficulties in connection with, 30.
Infinite series; quadrature of curves in, 247, 270, 293, 324.
Infinitesimal calculus. See Differential calculus.
Infinitesimals; use of, legitimate, 391.
Innocent IV. of Rome, 154.
Instruments, use of mathematical, objected to by Plato, 44, 46.
Integral calculus, 249, 310, 330, 360. See also Differential calculus.
Interference; principle of, 291, 406.
Interpolation, method of, 257, 265, 303, 330, 357, 376, 422.
Involutes, 362.
Involution; pencils in, 234.
Ionian school; the, 13-17, 21.
Ionians; history of mathematics begins with, 1.
Irrational magnitudes, 32-35, 53, 54, 55.
Isely; history of mathematics in Switzerland, ix.
Ishak ibn Honein, 140.
Isidorus, 109; ref. to, 124.
Isochronous curve, 310, 345.
Isoperimetrical figures; problem of, 345, 353, 365, 374.
Ivory, 408.

Jacobi, 417-9.
— ref. to, 391, 395, 410, 419, 420, 423, 424, 430.
Jamblichus, 110.
James II. of England, 301.
Jellett and Haughton; edition of MacCullagh, 435.
Jerome on finger symbolism, 112.
John Hispalensis, 159, ref. to, 161.

John XXII. of Rome, 185.
Jones (Bence), life of Faraday, [illegible].
Jordan, 4[illegible]; ref. to, [illegible].
Jordanus, 166.
Joule, [illegible].
Justinian, the emperor, 105.

Kaestner; hist. of mathematics, x.
[illegible]
[illegible]
Kempten, de, [illegible].
Kepler, 224-6;
ref. to, 30, 155, 162, 175, 221, 246, 248, 260, 264, 364.
Kepler's laws, 224, 225, 226, 246, 330, 414, [illegible].
[illegible] edition of the Arabian [illegible], [illegible].
[illegible]
[illegible]
[illegible]
[illegible]
[illegible], mathematical dictionary, x.
[illegible]
[illegible], 414; ref. to, [illegible].
[illegible]
[illegible], 414; ref. to, [illegible].
Kummer, 414;
ref. to, 362, [illegible], 420.

[illegible]
[illegible]
Lagrange, [illegible];
ref. to, [illegible].
[illegible]
[illegible]
[illegible]
[illegible]
Lambert, [illegible]; ref. to, [illegible].
[illegible]
Landen, [illegible]; ref. to, 391.
Laplace, [illegible];
ref. to, [illegible].
[illegible]

Latitude and longitude; introduction of, 81.
Latitude of Sparta found by Anaximander, 17.
Lavoisier, 390, 399.
Least action; principle of, 370, 374.
Least common multiple; Euclid's treatment of, 54.
Least squares; method of, 388, 393, 431.
Le Clerc on the medieval universities, 138.
Legendre, 391-5.
ref. to, 362, 365, 371, 379, 384, 385, 388, 391, 399, 403, 411, 416, 417, 419, 420, 422, 437.
Leibnitz, 338-44.
ref. to, 213, 235, 220, 243, 256, 295, 292, 293, 301, 295, 303, 304, 306, 316, 351, 356. See also Fluxional controversy.
Lemniscate; the, 42.
Lenses; construction of, 245, 269, 271, 272, 287, 299.
Leo VI. of Constantinople, 111.
Leo X. of Rome; Stifel on, 191.
Leonardo of Pisa, 159-162. ref. to, 197.
Letters used to denote magnitudes, 45, 149, 181, 192, 201, 207, 244, - describe diagrams, 19, 35.
Leverrier, 439-40. ref. to, 378.
Lewis; history of astronomy, x.
Lexell on Pappus' problem, 93.
L'Hospital, 318.
Lhuilier, 372. ref. to, 93.
Libri; ref. to, x, 188, 201, 218.
Lie, 414, 430.
Life assurance, 275.
Light; velocity of, 280, 407, 442, 444.
Lilavati; the, 145-8.
Lindemann; ref. to, 35, 432.
Lines of curvature, 396.
Linteria; the, 315.
Linus of Liège, 299.
Liouville, 414. ref. to, 428.
Lissajous on Foucault, 443.
Literal coefficients, 181, 204.
Logarithmic series; the, 275, 277, 314.
Logarithms; invention of, 174-6, 193.

Logarithms, tables of, 176.
Λογιστική; signification of, 52.
Lorentz; life of Alcuin, 124.
Louis XIV. of France, 250. 271. 272. 277. 339. 379.
Lucas of Liège, 290.
Lucas di Burgo. See Pacioli.
Lucian, 27.
Lunar tables, 352. 370. 386. 439.
Lunes; quadrature of, 37. 38.
Luther, 191. 192.

MacCullagh, 414. ref. to, 435. 441.
Maclaurin, 359–62.
— ref. to, 248. 335. 352. 363. 364. 365. 378. 392.
Macmahon, 414. 432.
Magic squares, 112. 113. 276. 351.
Magnetism, 295. 404. 405. 407. 421. 442. 444.
Magnifying power of a telescope, 278.
Marcellus, 60. 69.
Marie; hist. of mathematics, x.
Mariotte, 279.
Martin; ref. to, 81. 114. 218.
Mary of England, 191.
Mascheroni, 373.
Mass; centres of, 67. 93. 266.
— measurement of, 311.
Master in grammar; degree of, 135–6.
Master of arts; degree of, 131.
Mathematici Veteres; the, 106.
Mathematics; hist. of, ix. x, 12.
Maupertuis, 370.
Maurice of Orange, Prince, 217. 238.
Maurolycus, 201–2.
Maxima and minima; determination of, 265. 273. 277. 337. 341. 361.
Maxwell, 443–5. ref. to, 234.
Mayer, 367. 370.
Mechanical curves, 242.
Mechanics. Treatment by Archytas, 28; Aristotle, 45; Archimedes, 67–9; and Pappus, 93. Development of, by Stevinus and Galileo, chapter XIII, section 1. Huygens on, 271.2. Treated dynamically by Newton, chapter XVI. Subsequently extended by (among others) d'Alembert, Maclaurin, Euler, Lagrange, Laplace, and Poisson, chapters XVII. XVIII.
Melancthon, 180. 192.
Menæchmian triads; the, 43.
Menæchmus, 43–4.
— ref. to, 34. 48. 71.
Menelaus, 87. ref. to, 300.
Menou, General, 401.
Mercantile arithmetic, 160. 169. 170. 185–6. 187. 194.
Mercator, 277. ref. to, 203.
Méré, de, 253.
Mersenne, 237. ref. to, 250.
Metrodorus, 94.
Méziriac, 196–7. ref. to, 264. 377.
Microscope; invention of, 280.
Milo of Tarentum, 18. 20.
Minkowski, 427.
Minus. See Subtraction.
Mittag-Leffler, 414. ref. to, 436.
—; meanings of, 218.
— origin of sign, 185. 186. 192.
Möbius, 414. ref. to, 432.
Modern mathematics; features of, chapter XIV.
Mohammed; signature of, 10.
Mohammedan conquest of Egypt, 100.
Mollieres, 310.
Moments in theory of fluxions, 327.
Monge, 395–7.
— ref. to, 365. 432. 437.
Montague, 305.
Montmort, de, 349.
Montucla, 107.
— ref. to, x. 94. 228. 275. 278.
Moon; mass of, 320.
Morley; life of Cardan, 197.
Moschopulus, 112. ref. to, 351.
Motion; Newton's laws of, 310–12.
— ref. to, 219. 246. 372.
Mouton, 292. 389.
Müller. See Regiomontanus.
Mullinger; ref. to, 123. 124. 128.
Multiple points, 323. 348. 340.
Multiplication; processes of, 171. 172.
— symbols for, 211. 213.
— table, 121. 172.
Murdoch, 323.

Montucla, hist. of mathematics, x.
Müller, J., of Regiomontanus, 161.
Music, a subject in the quadrivium, 23, 24, 36, 104, 125, 126.
Mutawakkil, the caliph, 140.
Mydorge, 247; ref. to, 250.

Napier, John, 174-8.
ref. to, 172, 176, 177, 206, 240.
Napier, Mark, life of John Napier, 174.
Napoleon I., 377, 393, 400, 407, 402, 404, 405, 411.
Nasir Eddin, 141, 142, 157.
Naucrates, 51.
Nemorarius, Jordanus, 161.
Nebular hypothesis, 386.
Neil, 257.
Neo-Platonists; ref. to, x, 55, 61, 95, 116.
Neumann, 444; ref. to, 391, 441.
Newton; life and works of, and table of contents, chapter xvi.
ref. to, 51, 70, 79, 207, 210, 215, 220, 222, 225, 229, 232, 234, 237, 238, 239, 260, 262, 272, 274, 276, 278, 300, 312, 313, 330, 337, 353, 355, 356, 358, 360, 362, 363, 364, 365, 366, 367, 390, 401, 432. See also Principia.
Newton's rule, 288, 295, 382, 401.
Nicholas of Cusa, 162.
Nicholas V of Rome, 160, 165.
Nicolas, Paul, 162.
Nicole, 340.
Nicomachus, 82-4.
ref. to, 94, 97, 105, 107.
Nicomedes, 76.
Nicoteles, 73.
Nieuwentyt, 281.
Nines, casting out the, 154, 171.
Nixon, edition of Maxwell, 115.
Nizze, edition of Theodosius, 95.
Nonante; old French for ninety, 115.
Numbers; defective perfect and excessive, 37.
Eratosthenes, 257.
Pythagorean, 37, 82.
Numbers, theory of. Treatment of by Euclid, 54-5; Diophantus, 102; Mohammed, 151; Fermat, 261-5; Euler, 369; Lagrange, 377; Legendre, 393-4; and recent developments of by Gauss, Smith, and others, chapter xix, section 2.
Numerals; symbols for. Alexandrian, 120; Attic, 120; Egyptian, 119; Indian (or Arabic), 111, 147, 150, 159, 160, 161, 170; Roman, 119, 120, 170.
Numeration; systems of, chapter vii. See also Numerals.

Octante; old French for eighty, 115.
Oenopides of Chios, 20.
Offa, 121.
Oldenburg, 262, 323, 331, 332, 339.
Olleris; life of Gerbert, 128.
Oltalano on Pappus' problem, 98.
Omar, the caliph, 105.
Omega function, 410, 428.
Operations; calculus of, 357, 372.
Oppert; ref. to, 7.
Optics (geometrical), Euclid, 56; Pappus, 92; Alhazen, 155; Roger Bacon, 165; Snell, 218; Descartes, 246; Barrow, 269; Newton, 289, 290. Robert Smith, 358; Sir William Hamilton, 420,
(physical), 56, 273, 291, 400, 403, 407, 430, 441, 443, 445.
Oscillation; centre of, 272, 316, 350.
Osculating circle, 312.
Otho, 201.
Oughtred, 211-2.
ref. to, 203, 214, 281.
Oxford; the university of, 133, 134, 137, 151, 191, 256, 278, 279, 350, 424.
Ozanam, 107.

π; numerical values assigned for, 7, 9, 143, 144.
— approximations to or series for, 61, 146, 209, 210, 258, 271, 278.
— incommensurability of, 35, 278, 315, 371, 391.
— introduction of symbol, 307.
Pachymeres, 112.
Pacioli, 187-90.
— ref. to, 171, 172, 174, 192, 194, 195, 212, 213.

Paciolus. See Pacioli.
Pappus, 92–9.
— ref. to, 47, 49, 67, 70, 71, 77, 86, 202, 233, 242, 247, 305, 372.
Parabola; evolute of, 273.
— quadrature of, 62–3, 248, 257, 266.
— rectification of semicubical, 258.
Parallel lines, 89, 225, 227, 393.
Parallelogram of forces, 45, 217, 312.
Pardies, 290.
Parent, 318.
Paris; the university of, 138, 140, 151, 183, 189, 202, 275.
Pascal, 240–50.
— ref. to, 51, 207, 227, 237, 258, 260, 267, 271, 276, 277, 360, 381, 395, 433.
Peacock, 409.
— ref. to, 95, 111, 160, 163, 172, 332, 401, 408.
Pekhin, 360.
Peletier, 201.
Pell, 214.
Pemberton, 308.
Pendulum; motion of, 218, 221, 271, 272, 279, 311, 316.
Pentagrammatic, the, 19.
Pépin on Frénicle's problem, 270.
Perfect numbers, 27, 64.
Pericles, 33.
Perrault, 310.
Perseus, 78.
Petrarch, 141, 160.
Phaborinus, 47.
Phonnyches, 18.
Philip II. of Spain, 207.
Philolaus, 31, 43.
Philonides, 71.
Philoponus, 88.
Philosophiae; Newton on, 318.
Philosophy; successive views of, 240.
Phoenician mathematics, 3, 7, 10, 119.
Physics; mathematical, 234; chapter XVIII, section 3; and chapter XIX, section 8.
— history of, ix.
Picard, 287.
Piola; éloge on Cavalieri, 217.
Pitiscus, 201. ref. to, 208.
Plana, 414. ref. to, 439.
Planetary motions, 376, 379, 394, 395, 405, 409–10.
Planudes, 111, 170.
Plato, 39–40.
— ref. to, 24, 26, 29, 32, 33, 34, 41, 43, 44, 50, 75.
Plücker, 135.
Plus. See Addition.
— ; meanings of, 190, 239.
— origin of, 185, 190, 192.
Plutarch; ref. to, 14.
Poggendorff; dictionary of math., x.
Poinsot, 414.
— ref. to, 122, 336, 414.
Poisson, 404.
Point; Pythagorean def. of, 21.
Poisson, 403–5.
— ref. to, 365, 392, 414, 441.
Polarization of light, 370, 407.
Poles and polars, 337.
Polygonal numbers, 22, 27, 117.
Polygons; regular, capable of Euclidean construction, 421.
Polyhedral functions, 135.
Polyhedrons; the five regular, 21, 25, 44, 52, 73.
— the thirteen semi-regular, 65.
Poncelet, 392.
— ref. to, 93, 365, 122.
Pontécoulant, 414. ref. to, 130.
Porisms of Euclid, 56.
— of Diophantus, 102.
Port-Royal; society of, 250, 251.
Portsmouth papers of Newton, 288, 302, 307, 310.
Potential; the, 371, 405, 443.
Pott on numerals, 114.
Poudra on Desargues, 251.
Powers, 200, 211, 214, 215, 217, 241, 256, 324.
Predari; life of Cavalieri, 215.
Priestley, 290.
Prime and ultimate ratios, 213, 381.
Primes; Euclid's treatment of, 51.
Principia of Newton, 297–303, 308–31.
— ref. to, 231, 240, 252, 272, 280, 291, 292, 295, 303, 305, 306, 307, 313, 319, 325, 357, 420.

Rosen on Alkarismi's Al-gebr, 151.
Rosenhain, 414. ref. to, 419.
Rosenthal; math. encyclopædia, x.
Roulettes; tangents to, 243.
Routh on Newton, 315.
Royal Institution of London, 399.
Royal Society of London, 279.
Rudolph, 192.
Rudolph II. of Bohemia, 224.
Rumford, 399–400.

Saint-Mesme, marquis de, 348.
Saint-Venant, 414. ref. to, 441.
Saint-Vincent, 275.
— ref. to, 271. 277.
Salerno; the university of, 129. 130.
Salmon, 414. ref. to, 432. 435.
Sanderson's Logic, 284.
Sang, 176.
Sardou; life of Cardan, 197.
Saunderson of Cambridge, 334.
Saurin, 348.
Scaliger, 209.
Schering; edition of Gauss, 423.
Schneider on Roger Bacon, 163.
Schooten, 275. ref. to, 207. 284.
Schott, 210.
Schubert, 414. ref. to, 435.
Scratch system of division, 172–4.
Screw; the Archimedean, 59.
Screws; theory of, 433.
Secant; origin of term, 215.
Section; the golden, 41. 52.
Sedillot; ref. to, 2. 105. 139. 153. 155.
Septante; old French for seventy, 115.
Serenus, 87. 300.
Series; expansion in, 367.
Serret, 414. ref. to, 383. 432.
Sexagesimal division of angles; origin of, 6. 215.
Sextant; invention of, 292.
Sforza, 187.
Shakespeare; ref. to, 168.
Siddhanta of Brahmagupta, 143–4.
Signs; Descartes' rule of, 200. 244.
— rule of, in multiplication, 98.
Simple equations; methods of solution of, 95.
Simplicius, 38.
Simpson, Thomas, 367.
Simson, Robert, 49. ref. to, 73. 363.
Sine, 142. 154. 155. 180. 212. 215.
Sin x; series for, 278. 292. 344.
Sin^{-1} x; series for, 278. 292.
Sixtus IV. of Rome, 180.
Slee; life of Alcuin, 124.
Sluze, de, 280. ref. to, 270.
Smith, Henry, 424–8. ref. to, 419.
Smith, Robert, 358.
Snell, 218. ref. to, 245.
Socrates, 39.
Solid of least resistance, 307. 348.
Solids; the five regular, 21. 25. 44. 52. 78.
Solids; the thirteen semi-regular, 65.
Sophists; the, 32.
Sorbonne, 131.
Sound; Rayleigh on, 441.
— velocity of, 282. 319. 374. 390. 406.
Spedding on Francis Bacon, 222.
Sphere; surface and volume of, 64.
Spherical harmonics, 385. 392.
Spheroids; Archimedes on, 65.
Spinoza and Leibnitz, 341.
Spiral of Archimedes, 64.
Spiral; the equiangular, 345.
Sponius; edition of Cardan, 201.
Square root; symbol for, 192.
Squaring the circle. See Circle.
Statics. See Mechanics.
Staudt (von), 415. ref. to, 395. 433.
Steam-engine invented by Hero, 83.
Steichen on Stevinus, 210.
Stein on Leibnitz and Spinoza, 341.
Steiner, 415.
— ref. to, 395. 418. 432. 433.
Stevinus, 216–8.
— ref. to, 70. 170. 203. 204. 205. 214.
Stewart, Dugald, 303.
Stewart, Matthew, 363.
Stifel, 191–3.
— ref. to, 174. 202. 204. 213.
Stiffelius. See Stifel.
Stirling, 328.
Stobæus, 48.
Stokes, 136.
Stokes, 415. ref. to, 352. 441.
Strachey; ed. of the Bjita ganita, 145.
Stress; force one aspect of, 312.
String; vibrating, theory of, 347. 354. 355. 357. 374.
Sturm; ref. to, 372. 402.
Style or gnomon, 17.

Subtangent; curve whose, is constant, 295, 311.
Subtangent; determinations of, 205, 278, 279.
Subtraction; process of, 171.
 symbols for, 9, 106, 101, 138, 174, 195, 190, 189, 191, 192, 203, 213.
Suidas; ref. to, 16.
Sun-dials, 17.
Sun; distance and radius of, 57, 59.
Suter; hist. of mathematics, x, 85.
Swan-pan, 110-119. See Abacus.
Sylow and Lie; ed. of Abel, 417.
Sylvester II. (Gerbert), 120-3.
Sylvester, 416.
 ref. to, 336, 181, 188.
Symbolic algebra, 86, 201, 205.
Symbols; origin of algebraical, 192-5.
 trigonometrical, 215.
Symmetrical functions of roots of an equation, 201, 372, 189.
Syncopated algebra, 86, 115, 181, 185, 189.

Tabit ibn Korra, 111, 158.
Tables; lunar, 362, 270, 300, 199.
Tables of logarithms, 178.
 trigonometrical functions, 80, 90, 119, 111, 155, 180, 201.
Tangent; origin of term, 215.
Tan x; series for, 311.
Tangents to curves, 189, 241, 260, 275, 296, 267, 311.
Tannery; ref. to, x, 35, 102.
Tartaglia, 183-7.
 ref. to, 172, 187, 193, 189, 200, 207, 212, 213, 379.
Tartalea. See Tartaglia.
Tautochronous; cycloid is, 271.
Taylor (Brook), 284-5.
 ref. to, 317, 319, 324, 374.
 's theorem, 257, 301, 394.
Taylor, J.; ed. of the Lilavati, 145.
Tchebycheff, 125. ref. to, 424.
Telescopes, 230, 271, 252, 270, 295, 269.
Tension of elastic string, 270.
Terquem, 152.
Tetrad; the, 22, 24.
Text books; use of, not allowed by Pythagoras, 21.
Thales, 10-16. ref. to, 4.
Theaetetus, 34. 41. 50. 59.
Theano, 18.
Theodorus of Cyrene, 31, 19.
Theodosius, 95. ref. to, 259.
Theon of Alexandria, 101.
 ref. to, 49, 50.
Theon of Smyrna, 94.
Theophrastus, 13.
Thermodynamics, 102, 411.
Thermometer, invention of, 230, 270.
Theta function. See chapter xx, section 1; also p. 122.
Thick plates; colours of, 301.
Thin plates; colours of, 272, 301.
Thomson, Sir William, 115.
 ref. to, 281, 293, 390, 403, 411, 414.
Three bodies; problem of, 317, 350, 375, 376.
Thymaridas, 23. ref. to, 95.
Tides; theory of, 311, 317, 320, 363, 410.
Timaeus of Locri, 31.
Time; measurement of, 240.
Todhunter; ref. to, x, 362, 116.
Torricelli, 278.
 ref. to, 222, 250, 180.
Tortuous curves, 253, 384.
Τὸ τρισ ἄλογον ἀγόμενον, 19.
Trajectories, 305, 316.
Tremblay, 232.
Triangle; area of, 91, 82, 110, 114, 162.
 Pascal's arithmetical, 252.
Triangular numbers, 27.
Trigonometrical functions; constructions for, 212, 215.
 symbols; origin of, 215.
 tables, 80, 99, 142, 141, 155, 180, 201.
 terms; origin of, 155, 215.
Trigonometry. Ideas of in Rhind papyrus, 8, 9. Created by Hipparchus, 80. Considered a part of astronomy and treated as such by the Greeks and Arabs, 151. Hindus work on, 142, 149. Treated by most of the mathematicians of the renaissance, chapters xii, xiii. Development of by John Bernoulli, 316, and Euler, 357.